# Complex structures on indecomposable 6-dimensional nilpotent real Lie Algebras *

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**Abstract**

We compute all complex structures on indecomposable 6-dimensional real Lie algebras and their equivalence classes. We also give for each of them a global holomorphic chart on the connected simply connected Lie group associated to the real Lie algebra and write down the multiplication in that chart.

## 1 Introduction.

In the classification of nilmanifolds, a important question is to determine the set of all integrable left invariant complex structures on a given connected simply connected real nilpotent finite dimensional Lie group, or at the Lie algebra level the set $\mathfrak{X}_{\mathfrak{g}}$ of integrable complex structures on the nilpotent Lie algebra $\mathfrak{g}$, and its moduli space ([3],[2], [10]). In the case of 6-dimensional real nilpotent Lie groups, an upper bound has been given in [9] for the dimension of $\mathfrak{X}_{\mathfrak{g}}$, based on a subcomplex of the Dolbeault complex. These bounds are listed there, and Lie algebras which do not admit complex structures are specified. However, no detailed descriptions of the spaces $\mathfrak{X}_{\mathfrak{g}}$ are given. The aim of the present paper is to contribute in this area by supplying explicit computations of the various $\mathfrak{X}_{\mathfrak{g}}$ and their equivalence classes for any indecomposable 6-dimensional real Lie algebra $\mathfrak{g}$. We here are interested only in indecomposable Lie algebras, though direct products could be processed in the same way.

## 2 Preliminaries.

### 2.1 Labeling the algebras.

There are 22 indecomposable nonisomorphic nilpotent real 6-dimensional Lie algebras in the Morozov classification, labeled $M1$-$M22$ ([5]). Types $M14$ and $M18$ are splitted in $M14_{\pm 1}$ and $M18_{\pm 1}$. Over $\mathbb{C}$, types $M14$ and $M18$ are not splitted and types $M5$ and $M10$ do not appear. In [6], one is concerned with rank and weight systems over $\mathbb{C}$, and a different classification is used. The correspondance with Morozov types appears there on page 130. In the present paper, we label the algebras according to [6], except for $M5$, $M10$, $M14$ and $M18$. Note that $M5$ is the realification $\mathfrak{n}_{\mathbb{R}}$ of the 3-dimensional complex Heisenberg algebra $\mathfrak{n}$. Though $M10$ is not a realification, it appears as a subalgebra of the realification $(\mathfrak{g}_4)_{\mathbb{R}}$ of the complex 4-dimensional generic filiform Lie algebra $\mathfrak{g}_4$ in the isomorphic realisation $[a_1, a_2] = a_3, [a_1, a_3] = a_4, [a_2, a_3] = a_4$ : just take $x_1 = a_1, x_2 = ia_2, x_3 = ia_3, x_4 = a_3, x_5 = ia_4, x_6 = a_4$. Let $\mathfrak{g}$ be any of the labeled 6-dimensional real Lie algebras, and let $G_0$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$. From the commutation relations of the basis $(x_j)_{1\leqslant j\leqslant 6}$ of $\mathfrak{g}$ we use, the second kind canonical coordinates $(x \in G_0)$

$$x = \exp(x^1 x_1)\exp(y^1 x_2)\exp(x^2 x_3)\exp(y^2 x_4)\exp(x^3 x_5)\exp(y^3 x_6) \tag{1}$$

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yield a global chart for $G_0$ (see [11], Th. 3.18.11, p.243). We use this chart for $G_0$ in all cases but the case of $M5$ where the natural chart is used instead. For $1 \leqslant j \leqslant 6$, denote by $X_j$ the left invariant vector field on $G_0$ associated to $x_j$, *i.e.*

$$(X_j f)(x) = \left[\frac{d}{dt} f(x \exp(t x_j))\right]_{t=0} \quad \forall\, f \in C^\infty(G_0).$$

Then due to the commutation relations, we have in each case except $M5$ :

$$X_3 = \frac{\partial}{\partial x^2}\ ;\ X_4 = \frac{\partial}{\partial y^2}\ ;\ X_5 = \frac{\partial}{\partial x^3}\ ;\ X_6 = \frac{\partial}{\partial y^3}. \tag{2}$$

## 2.2 Complex structures.

Let $\mathfrak{g}$ any finite dimensional real Lie algebra, and let $G_0$ be the connected simply connected real Lie group with Lie algebra $\mathfrak{g}$. An almost complex structure on $\mathfrak{g}$ is a linear map $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -1$. An almost complex structure on $G_0$ is a tensor field $x \mapsto J_x$ which at every point $x \in G_0$ is an endomorphism of $T_x(G_0)$ such that $J_x^2 = -1$. By definition, the almost complex structure on $G_0$ is left (resp. right) invariant if $J_{ax} = (\hat{L_a})_x J_x$ (resp. $J_{xa} = (\hat{R_a})_x J_x$) for all $a, x \in G_0$, where $(\hat{L_a})_x J_x$ (resp. $(\hat{R_a})_x J_x$) is the endomorphism $(L_a)_{*x} \circ J_x \circ (L_{a^{-1}})_{*ax}$ (resp. $(R_a)_{*x} \circ J_x \circ (R_{a^{-1}})_{*xa}$) of $T_{ax}(G_0)$ (resp. $T_{xa}(G_0)$), with $L_a$ (resp. $R_a$) the left (resp. right) translation $x \mapsto ax$ (resp. $x \mapsto xa$) and $(.)_*$ the differential. For any almost complex structure $J$ on $\mathfrak{g}$ there is a unique left invariant almost complex $\hat{J}$ structure on $G_0$ such that $\hat{J}_e = J$ ($e$ is the identity of $G_0$), and one has $\hat{J}_a = (\hat{L_a})_{*e} J$ for all $a \in G_0$. It is easily seen that $\hat{J}$ is right invariant if and only if

$$J \circ ad\, X = ad\, X \circ J \quad \forall X \in \mathfrak{g},$$

that is $(\mathfrak{g}, J)$ is a *complex Lie algebra*. From the Newlander-Nirenberg theorem ([8]), $\hat{J}$ is *integrable*, that is $G_0$ can be given the structure of a complex manifold with the same underlying real structure and such that $\hat{J}$ is the canonical complex structure, if and only if the torsion tensor of $\hat{J}$ vanishes, i.e. :

$$[\hat{J}X, \hat{J}Y] - [X, Y] - \hat{J}[\hat{J}X, Y] - \hat{J}[X, \hat{J}Y] = 0$$

for all vector fields $X, Y$ on $G_0$. By left invariance, this is equivalent to

$$[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0 \quad \forall X, Y \in \mathfrak{g}. \tag{3}$$

By a complex structure on $\mathfrak{g}$, we'll mean an *integrable* almost complex structure on $\mathfrak{g}$, that is one satisfying (3).

Let $J$ a complex structure on $\mathfrak{g}$ and denote by $G$ the group $G_0$ endowed with the structure of complex manifold defined by $\hat{J}$. Then a smooth function $f : G_0 \to G_0$ is holomorphic if and only if its differential commutes with $\hat{J}$ ([4], Prop. 2.3 p. 123) : $\hat{J} \circ f_* = f_* \circ \hat{J}$. Hence left translations are holomorphic. Right translations are holomorphic, that is $G$ is a complex Lie group, if and only if $\hat{J}$ is right invariant, i.e. $(\mathfrak{g}, J)$ is a complex Lie algebra. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ splits as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$ where $\mathfrak{g}^{(1,0)} = \{X - iJX;\, X \in \mathfrak{g}\}$, $\mathfrak{g}^{(0,1)} = \{X + iJX;\, X \in \mathfrak{g}\}$. We will denote $\mathfrak{g}^{(1,0)}$ by $\mathfrak{m}$. The integrability of $J$ amounts to $\mathfrak{m}$ being a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. In that way the set of complex structures on $\mathfrak{g}$ can be identified with the set of all complex subalgebras $\mathfrak{m}$ of $\mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{m} \oplus \bar{\mathfrak{m}}$, bar denoting conjugation in $\mathfrak{g}_{\mathbb{C}}$. This is the algebraic approach. Our approach is more trivial since we simply fix a basis of $\mathfrak{g}$ and compute all possible matrices in that basis for a complex structure. From now on, we'll use the same notation $J$ for $J$ and $\hat{J}$ as well. For any $x \in G_0$, the complexification $T_x(G_0)_{\mathbb{C}}$ of the tangent space also splits as the direct sum of the holomorphic vectors $T_x(G_0)^{(1,0)} = \{X - iJX;\, X \in T_x(G_0)\}$ and the antiholomorphic vectors $T_x(G_0)^{(0,1)} = \{X + iJX;\, X \in T_x(G_0)\}$. Let $H_{\mathbb{C}}(G)$ be the space of complex valued holomorphic functions on $G$. Then $H_{\mathbb{C}}(G)$ is comprised of all complex smooth functions $f$ on $G_0$ which are annihilated by any antiholomorphic vector field. This is equivalent to $f$ being annihilated by all

$$\tilde{X}_j^- = X_j + iJX_j \quad 1 \leqslant j \leqslant n \tag{4}$$

with $(X_j)_{1\leqslant j\leqslant n}$ the left invariant vector fields associated to a basis $(x_j)_{1\leqslant j\leqslant n}$ of $\mathfrak{g}$. Hence :

$$H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \ \forall j\ \ 1 \leqslant j \leqslant n\}. \tag{5}$$

Finally, the automorphism group Aut $\mathfrak{g}$ of $\mathfrak{g}$ acts on the set $\mathfrak{X}_\mathfrak{g}$ of all complex structures on $\mathfrak{g}$ by $J \mapsto \Phi^{-1} \circ J \circ \Phi \quad \forall \Phi \in \text{Aut } \mathfrak{g}$. Two complex structures $J_1, J_2$ on $\mathfrak{g}$ are said to be *equivalent* if they are on the same Aut $\mathfrak{g}$ orbit.

### 2.3 Presentation of results.

We consider here only indecomposable 6-dimensional nilpotent real Lie algebras which admit complex structures. For each such $\mathfrak{g}$, we first give the commutation relations of the basis $(x_j)_{1\leqslant j\leqslant 6}$ of $\mathfrak{g}$ we use, and the matrices $J = (J_j^k) = (\xi_j^k)$ in that basis of the elements of $\mathfrak{X}_\mathfrak{g}$. The parameters are 'boxed'. These matrices have been obtained by developping specific programs with the computer algebra system *Reduce* by A. Hearn. The programs solve simultaneously the equation $J^2 = -1$ and the torsion equations $ij|k$ ($1 \leqslant i, j, k \leqslant 6$ ) obtained by projecting on $x_k$ the equation $[Jx_i, Jx_j] - [x_i, x_j] - J[Jx_i, x_j] - J[x_i, Jx_j] = 0$. We do not enter computational technicalities here, referring instead to the technical report [7]. Let's simply say that the equations are *semilinear* in the sense that they can be solved in a succession of steps, each of which consists in solving some equation of degree 1 in some variable. For all the Lie algebras we consider, we prove that $\mathfrak{X}_\mathfrak{g}$ is a (smooth) submanifold of $\mathbb{R}^{36}$. The dimension of $\mathfrak{X}_\mathfrak{g}$ is equal to the upper bound given in [9] except in the case of $M10$. Then we give the automorphism group of $\mathfrak{g}$, representatives of the various equivalence classes, and the commutation relations of the corresponding algebra $\mathfrak{m} = \mathfrak{g}^{(1,0)}$ in terms of the basis $(\tilde{x}_j)_{1\leqslant j\leqslant 6}$ with $\tilde{x}_j = x_j - iJx_j$. As $\mathfrak{m}$ is a 3-dimensional complex Lie algebra, it is either abelian or isomorphic to the complex Heisenberg Lie algebra $\mathfrak{n}$. Hence, as a real Lie algebra, $\mathfrak{m}$ is either abelian or isomorphic to $M5$. Finally, we compute the left invariant vector fields $X_1, X_2$ on $G_0$ in terms of the second kind canonical coordinates (1) (except in the case of $M5$, the 4 others appear in (2)), a global holomorphic chart on $G$ and the explicit look of the multiplication in $G$ in terms of that chart. The fact that left translations are holomorphic, though the multiplication isn't except for the canonical structure on $M5$, appears clearly on these formulae. There we make use of the following formula which is easily checked from the Campbell-Hausdorff-Baker Formula :

$$e^X e^Y = e^{[X,Y]+\frac{1}{2}\,([X,[X,Y]]+[Y,[X,Y]])+\frac{1}{6}\,([X,[X,[X,Y]]]+[Y,[Y,[X,Y]]])+\frac{1}{4}\,[X,[Y,[X,Y]]] \bmod \mathcal{C}^5\mathfrak{g}}\, e^Y e^X \tag{6}$$

where $\mathcal{C}^5\mathfrak{g}$ denotes the $5^{\text{th}}$ central derivative. As usual, for complex $z = x + iy$ $(x, y \in \mathbb{R})$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$ ; $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$. We'll abbreviate 'complex structure' to CS.

# 3 Lie Algebra $\mathcal{G}_{6,3}$ (isomorphic to $M3$).

Commutation relations for $\mathcal{G}_{6,3}$ : $[x_1, x_2] = x_4$; $[x_1, x_3] = x_5$; $[x_2, x_3] = x_6$.

### 3.1 Case $\xi_6^1 \neq 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & \boxed{\xi_3^1} & \xi_6^3 & -\xi_6^2 & \boxed{\xi_6^1} \\ * & (-\xi_6^5\xi_6^2 - \xi_5^5\xi_6^1 + \xi_6^2\xi_2^1)/\xi_6^1 & (\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1 + \xi_6^2\xi_3^1)/\xi_6^1 & \xi_6^3\xi_6^2/\xi_6^1 & -{\xi_6^2}^2/\xi_6^1 & \boxed{\xi_6^2} \\ * & * & (\xi_6^5\xi_6^2 + \xi_5^5\xi_6^1 + \xi_6^3\xi_3^1)/\xi_6^1 & {\xi_6^3}^2/\xi_6^1 & -\xi_6^3\xi_6^2/\xi_6^1 & \boxed{\xi_6^3} \\ * & (\xi_6^5\xi_3^1 - \xi_3^5\xi_6^1 + \xi_6^4\xi_2^1)/\xi_6^1 & * & (-\xi_6^5\xi_6^2 - \xi_5^5\xi_6^1 + \xi_6^4\xi_6^3)/\xi_6^1 & \boxed{\xi_5^4} & \boxed{\xi_6^4} \\ * & \boxed{\xi_2^5} & \boxed{\xi_3^5} & * & \boxed{\xi_5^5} & \boxed{\xi_6^5} \\ * & * & * & * & * & * \end{pmatrix} \tag{7}$$

where $\mathbf{J_1^2} = (\xi_6^5{\xi_6^2}^2 + \xi_5^5\xi_6^2\xi_6^1 - \xi_6^4\xi_6^3\xi_6^2 - \xi_5^4\xi_6^3\xi_6^1 + \xi_6^2\xi_6^1\xi_1^1)/{\xi_6^1}^2$; $\mathbf{J_1^3} = ({\xi_6^5}^2{\xi_6^2}^3 + 2\xi_6^5\xi_5^5{\xi_6^2}^2\xi_6^1 - \xi_6^5\xi_6^4\xi_6^3{\xi_6^2}^2 - \xi_6^5\xi_5^4\xi_6^3\xi_6^2\xi_6^1 + {\xi_5^5}^2\xi_6^2{\xi_6^1}^2 - \xi_5^5\xi_6^4\xi_6^3\xi_6^2\xi_6^1 - \xi_5^5\xi_5^4\xi_6^3{\xi_6^1}^2 + \xi_6^4\xi_6^3\xi_6^2\xi_6^1\xi_1^1 + \xi_5^4\xi_6^3{\xi_6^1}^2\xi_1^1 + \xi_6^2{\xi_6^1}^2)/({\xi_6^1}^2(\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1))$; $\mathbf{J_2^3} = (-{\xi_6^5}^2{\xi_6^2}^2 - 2\xi_6^5\xi_5^5\xi_6^2\xi_6^1 - {\xi_5^5}^2{\xi_6^1}^2 + \xi_6^4\xi_6^3\xi_6^2\xi_2^1 + \xi_5^4\xi_6^3\xi_6^1\xi_2^1 - {\xi_6^1}^2)/(\xi_6^1(\xi_6^4\xi_6^2 + \xi_5^4\xi_6^1))$; $\mathbf{J_1^4} = (-{\xi_6^5}^3\xi_5^4{\xi_6^2}^2\xi_6^1 - {\xi_6^5}^3{\xi_6^2}^3\xi_3^1 + {\xi_6^5}^2\xi_5^5\xi_6^4{\xi_6^2}^2\xi_6^1 -$

$2{\xi^5_6}^2\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^2 - 2{\xi^5_6}^2\xi^5_5{\xi^2_6}^2\xi^1_6\xi^1_3 + {\xi^5_6}^2\xi^5_3{\xi^2_6}^3\xi^1_6 + 2{\xi^5_6}^2\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_3 + {\xi^5_6}^2\xi^4_6{\xi^2_6}^2\xi^1_6\xi^1_1 + 2{\xi^5_6}^2\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_3 + 2\xi^5_6{\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^2 - \xi^5_6{\xi^5_5}^2\xi^4_5{\xi^1_6}^3 -$
$\xi^5_6{\xi^5_5}^2\xi^2_6{\xi^1_6}^2\xi^1_3 + 2\xi^5_6\xi^5_5\xi^5_3{\xi^2_6}^2{\xi^1_6}^2 + 2\xi^5_6\xi^5_5\xi^4_6\xi^3_6\xi^2_6\xi^1_6\xi^1_3 + 2\xi^5_6\xi^5_5\xi^4_6\xi^2_6{\xi^1_6}^2\xi^1_1 + 2\xi^5_6\xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^2\xi^1_3 - 2\xi^5_6\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_6\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 +$
$\xi^5_6{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_2 + 2\xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_2 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^2\xi^1_2 - \xi^5_6\xi^4_5{\xi^1_6}^3 - \xi^5_6\xi^2_6{\xi^1_6}^2\xi^1_3 + {\xi^5_5}^3\xi^4_6{\xi^1_6}^3 + {\xi^5_5}^2\xi^5_3\xi^2_6{\xi^1_6}^3 + {\xi^5_5}^2\xi^4_6{\xi^1_6}^3\xi^1_1 - 2\xi^5_5\xi^5_3\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^2 -$
$2\xi^5_5\xi^5_3\xi^4_5\xi^3_6{\xi^1_6}^3 + \xi^5_5\xi^4_6{\xi^1_6}^3 + \xi^5_3\xi^2_6{\xi^1_6}^3 - \xi^5_2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 - \xi^5_2{\xi^4_5}^2\xi^3_6{\xi^1_6}^3 + \xi^4_6{\xi^1_6}^3\xi^1_1)/({\xi^1_6}^2((\xi^5_6\xi^2_6 + \xi^5_5\xi^1_6)^2 + {\xi^1_6}^2));$
$\mathbf{J^4_3} = (-{\xi^5_6}^2\xi^4_6{\xi^2_6}^2\xi^1_3 - 2{\xi^5_6}^2\xi^4_5\xi^2_6\xi^1_6\xi^1_3 - 2\xi^5_6\xi^5_5\xi^4_5{\xi^1_6}^2\xi^1_3 + 2\xi^5_6\xi^5_3\xi^4_6{\xi^2_6}^2\xi^1_6 + 2\xi^5_6\xi^5_3\xi^4_5\xi^2_6{\xi^1_6}^2 - \xi^5_6{\xi^4_6}^2{\xi^2_6}^2\xi^1_2 - 2\xi^5_6\xi^4_6\xi^4_5\xi^2_6\xi^1_6\xi^1_2 -$
$\xi^5_6{\xi^4_5}^2{\xi^1_6}^2\xi^1_2 + {\xi^5_5}^2\xi^4_6{\xi^1_6}^2\xi^1_3 + 2\xi^5_5\xi^5_3\xi^4_6\xi^2_6{\xi^1_6}^2 + 2\xi^5_5\xi^5_3\xi^4_5{\xi^1_6}^3 + \xi^5_2{\xi^4_6}^2{\xi^2_6}^2\xi^1_6 + 2\xi^5_2\xi^4_6\xi^4_5\xi^2_6{\xi^1_6}^2 + \xi^5_2{\xi^4_5}^2{\xi^1_6}^3 + \xi^4_6{\xi^1_6}^2\xi^1_3)/(\xi^1_6((\xi^5_6\xi^2_6 +$
$\xi^5_5\xi^1_6)^2 + {\xi^1_6}^2));$ $\mathbf{J^5_1} = (-{\xi^5_6}^2\xi^4_6\xi^4_5{\xi^2_6}^2\xi^1_6 - {\xi^5_6}^2{\xi^4_5}^2\xi^2_6{\xi^1_6}^2 + \xi^5_6\xi^5_5{\xi^4_6}^2{\xi^2_6}^2\xi^1_6 - \xi^5_6\xi^5_5{\xi^4_5}^2{\xi^1_6}^3 + \xi^5_6{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_3 + \xi^5_6{\xi^4_6}^2{\xi^2_6}^3\xi^1_2 +$
$\xi^5_6{\xi^4_6}^2{\xi^2_6}^2\xi^1_6\xi^1_1 + 2\xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_3 + 2\xi^5_6\xi^4_6\xi^4_5{\xi^2_6}^2\xi^1_6\xi^1_2 + 2\xi^5_6\xi^4_6\xi^4_5\xi^2_6{\xi^1_6}^2\xi^1_1 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^2\xi^1_3 + \xi^5_6{\xi^4_5}^2\xi^2_6{\xi^1_6}^2\xi^1_2 + \xi^5_6{\xi^4_5}^2{\xi^1_6}^3\xi^1_1 + {\xi^5_5}^2{\xi^4_6}^2\xi^2_6{\xi^1_6}^2 +$
${\xi^5_5}^2\xi^4_6\xi^4_5{\xi^1_6}^3 - \xi^5_3{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_3\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 - \xi^5_3{\xi^4_5}^2\xi^3_6{\xi^1_6}^3 - \xi^5_2{\xi^4_6}^2{\xi^2_6}^3\xi^1_6 - 2\xi^5_2\xi^4_6\xi^4_5{\xi^2_6}^2{\xi^1_6}^2 - \xi^5_2{\xi^4_5}^2\xi^2_6{\xi^1_6}^3 + {\xi^4_6}^2\xi^2_6{\xi^1_6}^2 +$
$\xi^4_6\xi^4_5{\xi^1_6}^3)/({\xi^1_6}^2(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6)^2);$ $\mathbf{J^5_4} = (-{\xi^5_6}^2{\xi^2_6}^2 - 2\xi^5_6\xi^5_5\xi^2_6\xi^1_6 + \xi^5_6\xi^4_6\xi^3_6\xi^2_6 + \xi^5_6\xi^4_5\xi^3_6\xi^1_6 - {\xi^5_5}^2{\xi^1_6}^2 - {\xi^1_6}^2)/(\xi^1_6(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$
$\mathbf{J^6_1} = (-{\xi^5_6}^4\xi^4_5{\xi^2_6}^4\xi^1_6 - {\xi^5_6}^4{\xi^2_6}^5\xi^1_3 + {\xi^5_6}^3\xi^5_5\xi^4_6{\xi^2_6}^4\xi^1_6 - 3{\xi^5_6}^3\xi^5_5\xi^4_5{\xi^2_6}^3{\xi^1_6}^2 - 4{\xi^5_6}^3\xi^5_5{\xi^2_6}^4\xi^1_6\xi^1_3 + {\xi^5_6}^3\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6 + 3{\xi^5_6}^3\xi^4_6\xi^3_6{\xi^2_6}^4\xi^1_3 +$
${\xi^5_6}^3\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_1 + {\xi^5_6}^3{\xi^4_5}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 + 3{\xi^5_6}^3\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3 + {\xi^5_6}^3\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_1 + 3{\xi^5_6}^2{\xi^5_5}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2 - 3{\xi^5_6}^2{\xi^5_5}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3 - 6{\xi^5_6}^2{\xi^5_5}^2{\xi^2_6}^3{\xi^1_6}^2\xi^1_3 -$
${\xi^5_6}^2\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6 + {\xi^5_6}^2\xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 + 7{\xi^5_6}^2\xi^5_5\xi^4_6\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3 - {\xi^5_6}^2\xi^5_5\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_2 + 2{\xi^5_6}^2\xi^5_5\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_1 + 2{\xi^5_6}^2\xi^5_5{\xi^4_5}^2\xi^3_6\xi^2_6{\xi^1_6}^3 +$
$7{\xi^5_6}^2\xi^5_5\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 - {\xi^5_6}^2\xi^5_5\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_2 + 2{\xi^5_6}^2\xi^5_5\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 - 2{\xi^5_6}^2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^4\xi^1_6 - 2{\xi^5_6}^2\xi^5_3\xi^4_5\xi^3_6{\xi^2_6}^3{\xi^1_6}^2 - {\xi^5_6}^2\xi^5_2\xi^4_6{\xi^2_6}^5\xi^1_6 -$
${\xi^5_6}^2\xi^5_2\xi^4_5{\xi^2_6}^4{\xi^1_6}^2 - 2{\xi^5_6}^2{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^3\xi^1_3 + {\xi^5_6}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^4\xi^1_2 - {\xi^5_6}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_1 - 4{\xi^5_6}^2\xi^4_6\xi^4_5{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3 + 2{\xi^5_6}^2\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_2 -$
${\xi^5_6}^2\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_1 - {\xi^5_6}^2\xi^4_6\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_3\xi^1_1 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^4\xi^1_6\xi^1_2\xi^1_1 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2{\xi^1_1}^2 - 2{\xi^5_6}^2{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_3 + {\xi^5_6}^2{\xi^4_5}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 -$
${\xi^5_6}^2\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3\xi^1_1 - {\xi^5_6}^2\xi^4_5{\xi^2_6}^3{\xi^1_6}^2\xi^1_2\xi^1_1 - {\xi^5_6}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3{\xi^1_1}^2 - 2{\xi^5_6}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3 - 2{\xi^5_6}^2{\xi^2_6}^3{\xi^1_6}^2\xi^1_3 + 3\xi^5_6{\xi^5_5}^3\xi^4_6{\xi^2_6}^2{\xi^1_6}^3 - \xi^5_6{\xi^5_5}^3\xi^4_5\xi^2_6{\xi^1_6}^4 -$
$4\xi^5_6{\xi^5_5}^3{\xi^2_6}^2{\xi^1_6}^3\xi^1_3 - 2\xi^5_6{\xi^5_5}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2 - \xi^5_6{\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3 + 5\xi^5_6{\xi^5_5}^2\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 - 2\xi^5_6{\xi^5_5}^2\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_2 + \xi^5_6{\xi^5_5}^2\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 +$
$\xi^5_6{\xi^5_5}^2{\xi^4_5}^2\xi^3_6{\xi^1_6}^4 + 5\xi^5_6{\xi^5_5}^2\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - 2\xi^5_6{\xi^5_5}^2\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_6{\xi^5_5}^2\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_1 - 4\xi^5_6\xi^5_5\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^3{\xi^1_6}^2 - 4\xi^5_6\xi^5_5\xi^5_3\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_6\xi^5_5\xi^5_2\xi^4_6{\xi^2_6}^4{\xi^1_6}^2 - 2\xi^5_6\xi^5_5\xi^5_2\xi^4_5{\xi^2_6}^3{\xi^1_6}^3 - 2\xi^5_6\xi^5_5{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3 + 2\xi^5_6\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^3\xi^1_6\xi^1_2 - 2\xi^5_6\xi^5_5{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_1 - 4\xi^5_6\xi^5_5\xi^4_6\xi^4_5{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_3 +$
$4\xi^5_6\xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - 2\xi^5_6\xi^5_5\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^3{\xi^1_6}^2\xi^1_2\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3{\xi^1_1}^2 + \xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_6\xi^5_5{\xi^4_5}^2{\xi^3_6}^2{\xi^1_6}^3\xi^1_3 + 2\xi^5_6\xi^5_5{\xi^4_5}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - 2\xi^5_6\xi^5_5\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - 2\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 - 3\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4 -$
$4\xi^5_6\xi^5_5{\xi^2_6}^2{\xi^1_6}^3\xi^1_3 + 2\xi^5_6\xi^5_3{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^3\xi^1_6 + 4\xi^5_6\xi^5_3\xi^4_6\xi^4_5{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 2\xi^5_6\xi^5_3{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 - \xi^5_6{\xi^4_6}^3{\xi^3_6}^2{\xi^2_6}^3\xi^1_2 - 3\xi^5_6{\xi^4_6}^2\xi^4_5{\xi^3_6}^2{\xi^2_6}^2\xi^1_6\xi^1_2 -$
$3\xi^5_6\xi^4_6{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^2\xi^1_2 + \xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3 + 3\xi^5_6\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_3 + \xi^5_6\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_1 - \xi^5_6{\xi^4_5}^3{\xi^3_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^4 + 3\xi^5_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 +$
$\xi^5_6\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_1 + {\xi^5_5}^4\xi^4_6\xi^2_6{\xi^1_6}^4 - {\xi^5_5}^4\xi^2_6{\xi^1_6}^4\xi^1_3 - {\xi^5_5}^3{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3 - {\xi^5_5}^3\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4 + {\xi^5_5}^3\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - {\xi^5_5}^3\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + {\xi^5_5}^3\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3 -$
${\xi^5_5}^3\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2 - 2{\xi^5_5}^2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 - 2{\xi^5_5}^2\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^4 - {\xi^5_5}^2\xi^5_2\xi^4_6{\xi^2_6}^3{\xi^1_6}^3 - {\xi^5_5}^2\xi^5_2\xi^4_5{\xi^2_6}^2{\xi^1_6}^4 + {\xi^5_5}^2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - {\xi^5_5}^2{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 +$
$2{\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - {\xi^5_5}^2\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_1 - {\xi^5_5}^2\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - {\xi^5_5}^2\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 + {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^4 + {\xi^5_5}^2{\xi^4_5}^2\xi^3_6{\xi^1_6}^4\xi^1_2 -$
${\xi^5_5}^2\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3\xi^1_1 - {\xi^5_5}^2\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2\xi^1_1 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^5{\xi^1_1}^2 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^5 - 2{\xi^5_5}^2\xi^2_6{\xi^1_6}^4\xi^1_3 + 2\xi^5_5\xi^5_3{\xi^4_6}^2{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 4\xi^5_5\xi^5_3\xi^4_6\xi^4_5{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 +$
$2\xi^5_5\xi^5_3{\xi^4_5}^2{\xi^3_6}^2{\xi^1_6}^4 - \xi^5_5{\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3 - \xi^5_5\xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4 + \xi^5_5\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3 - \xi^5_5\xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2 + \xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3 - \xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2 - 2\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2{\xi^1_6}^3 -$
$2\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^4 + \xi^5_2{\xi^4_6}^3{\xi^3_6}^2{\xi^2_6}^3\xi^1_6 + 3\xi^5_2{\xi^4_6}^2\xi^4_5{\xi^3_6}^2{\xi^2_6}^2{\xi^1_6}^2 + 3\xi^5_2\xi^4_6{\xi^4_5}^2{\xi^3_6}^2\xi^2_6{\xi^1_6}^3 - \xi^5_2\xi^4_6{\xi^2_6}^3{\xi^1_6}^3 + \xi^5_2{\xi^4_5}^3{\xi^3_6}^2{\xi^1_6}^4 - \xi^5_2\xi^4_5{\xi^2_6}^2{\xi^1_6}^4 +$
${\xi^4_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_6}^2\xi^1_2 - {\xi^4_6}^2\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_1 + 2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_2 - \xi^4_6\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_1 - \xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^3\xi^1_3\xi^1_1 - \xi^4_6{\xi^2_6}^2{\xi^1_6}^3\xi^1_2\xi^1_1 - \xi^4_6\xi^2_6{\xi^1_6}^4{\xi^1_1}^2 + {\xi^4_5}^2\xi^3_6{\xi^1_6}^4\xi^1_2 -$
$\xi^4_5\xi^3_6{\xi^1_6}^4\xi^1_3\xi^1_1 - \xi^4_5\xi^2_6{\xi^1_6}^4\xi^1_2\xi^1_1 - \xi^4_5{\xi^1_6}^5{\xi^1_1}^2 - \xi^4_5{\xi^1_6}^5 - \xi^2_6{\xi^1_6}^4\xi^1_3)/({\xi^1_6}^3(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6)((\xi^5_6\xi^2_6 + \xi^5_5\xi^1_6)^2 + {\xi^1_6}^2));$ $\mathbf{J^6_2} = ({\xi^5_6}^2{\xi^2_6}^2\xi^1_3 +$
$2\xi^5_6\xi^5_5\xi^2_6\xi^1_6\xi^1_3 - \xi^5_6\xi^4_6\xi^3_6\xi^2_6\xi^1_3 + \xi^5_6\xi^4_6{\xi^2_6}^2\xi^1_2 - \xi^5_6\xi^4_5\xi^3_6\xi^1_6\xi^1_3 + \xi^5_6\xi^4_5\xi^2_6\xi^1_6\xi^1_2 + {\xi^5_5}^2{\xi^1_6}^2\xi^1_3 + \xi^5_5\xi^4_6\xi^2_6\xi^1_6\xi^1_2 + \xi^5_5\xi^4_5{\xi^1_6}^2\xi^1_2 + \xi^5_3\xi^4_6\xi^3_6\xi^2_6\xi^1_6 + \xi^5_3\xi^4_5\xi^3_6{\xi^1_6}^2 +$
$\xi^5_2\xi^4_6{\xi^2_6}^2\xi^1_6 + \xi^5_2\xi^4_5\xi^2_6{\xi^1_6}^2 - {\xi^4_6}^2\xi^3_6\xi^2_6\xi^1_2 - \xi^4_6\xi^4_5\xi^3_6\xi^1_6\xi^1_2 - \xi^4_6\xi^3_6\xi^2_6\xi^1_3\xi^1_2 - \xi^4_6{\xi^2_6}^2{\xi^1_2}^2 - \xi^4_6\xi^2_6\xi^1_6\xi^1_2\xi^1_1 - \xi^4_5\xi^3_6\xi^1_6\xi^1_3\xi^1_2 - \xi^4_5\xi^2_6\xi^1_6{\xi^1_2}^2 - \xi^4_5{\xi^1_6}^2\xi^1_2\xi^1_1 +$
${\xi^1_6}^2\xi^1_3)/({\xi^1_6}^2(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$ $\mathbf{J^6_3} = (-{\xi^5_6}^3{\xi^2_6}^3\xi^1_3 - 3{\xi^5_6}^2\xi^5_5{\xi^2_6}^2\xi^1_6\xi^1_3 + {\xi^5_6}^2\xi^5_3{\xi^2_6}^3\xi^1_6 + {\xi^5_6}^2\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_3 - {\xi^5_6}^2\xi^4_6{\xi^2_6}^3\xi^1_2 + 2{\xi^5_6}^2\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_3 -$
${\xi^5_6}^2\xi^4_5{\xi^2_6}^2\xi^1_6\xi^1_2 - {\xi^5_6}^2\xi^3_6{\xi^2_6}^2{\xi^1_3}^2 - {\xi^5_6}^2{\xi^2_6}^3\xi^1_3\xi^1_2 - {\xi^5_6}^2{\xi^2_6}^2\xi^1_6\xi^1_3\xi^1_1 - 3\xi^5_6{\xi^5_5}^2\xi^2_6{\xi^1_6}^2\xi^1_3 + 2\xi^5_6\xi^5_5\xi^5_3{\xi^2_6}^2{\xi^1_6}^2 - 2\xi^5_6\xi^5_5\xi^4_6{\xi^2_6}^2\xi^1_6\xi^1_2 + 2\xi^5_6\xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^2\xi^1_3 -$
$2\xi^5_6\xi^5_5\xi^4_5\xi^2_6{\xi^1_6}^2\xi^1_2 - 2\xi^5_6\xi^5_5\xi^3_6\xi^2_6\xi^1_6{\xi^1_3}^2 - 2\xi^5_6\xi^5_5{\xi^2_6}^2\xi^1_6\xi^1_3\xi^1_2 - 2\xi^5_6\xi^5_5\xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_1 - 2\xi^5_6\xi^5_3\xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_6\xi^5_3\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 + \xi^5_6{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_2 +$
$2\xi^5_6\xi^4_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_2 + \xi^5_6{\xi^4_5}^2\xi^3_6{\xi^1_6}^2\xi^1_2 - \xi^5_6\xi^2_6{\xi^1_6}^2\xi^1_3 - {\xi^5_5}^3{\xi^1_6}^3\xi^1_3 + {\xi^5_5}^2\xi^5_3\xi^2_6{\xi^1_6}^3 - {\xi^5_5}^2\xi^4_6\xi^3_6{\xi^1_6}^2\xi^1_3 - {\xi^5_5}^2\xi^4_6\xi^2_6{\xi^1_6}^2\xi^1_2 - {\xi^5_5}^2\xi^4_5{\xi^1_6}^3\xi^1_2 - {\xi^5_5}^2\xi^3_6{\xi^1_6}^2{\xi^1_3}^2 -$
${\xi^5_5}^2\xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_2 - {\xi^5_5}^2{\xi^1_6}^3\xi^1_3\xi^1_1 - 2\xi^5_5\xi^5_3\xi^4_6\xi^3_6\xi^2_6{\xi^1_6}^2 - 2\xi^5_5\xi^5_3\xi^4_5\xi^3_6{\xi^1_6}^3 - \xi^5_5{\xi^1_6}^3\xi^1_3 + \xi^5_3\xi^2_6{\xi^1_6}^3 - \xi^5_2{\xi^4_6}^2\xi^3_6{\xi^2_6}^2\xi^1_6 - 2\xi^5_2\xi^4_6\xi^4_5\xi^3_6\xi^2_6{\xi^1_6}^2 - \xi^5_2{\xi^4_5}^2\xi^3_6{\xi^1_6}^3 -$
$\xi^4_6\xi^3_6{\xi^1_6}^2\xi^1_3 - \xi^4_6\xi^2_6{\xi^1_6}^2\xi^1_2 - \xi^4_5{\xi^1_6}^3\xi^1_2 - \xi^3_6{\xi^1_6}^2{\xi^1_3}^2 - \xi^2_6{\xi^1_6}^2\xi^1_3\xi^1_2 - {\xi^1_6}^3\xi^1_3\xi^1_1)/({\xi^1_6}^2((\xi^5_6\xi^2_6 + \xi^5_5\xi^1_6)^2 + {\xi^1_6}^2));$ $\mathbf{J^6_4} = (-{\xi^5_6}^2{\xi^2_6}^3 - 2\xi^5_6\xi^5_5{\xi^2_6}^2\xi^1_6 +$
$2\xi^5_6\xi^4_6\xi^3_6{\xi^2_6}^2 + 2\xi^5_6\xi^4_5\xi^3_6\xi^2_6\xi^1_6 - {\xi^5_5}^2\xi^2_6{\xi^1_6}^2 + \xi^5_5\xi^4_6\xi^3_6\xi^2_6\xi^1_6 + \xi^5_5\xi^4_5\xi^3_6{\xi^1_6}^2 - {\xi^4_6}^2{\xi^3_6}^2\xi^2_6 - \xi^4_6\xi^4_5{\xi^3_6}^2\xi^1_6 - \xi^4_6{\xi^3_6}^2\xi^2_6\xi^1_3 - \xi^4_6\xi^3_6{\xi^2_6}^2\xi^1_2 - \xi^4_6\xi^3_6\xi^2_6\xi^1_6\xi^1_1 -$
$\xi^4_5{\xi^3_6}^2\xi^1_6\xi^1_3 - \xi^4_5\xi^3_6\xi^2_6\xi^1_6\xi^1_2 - \xi^4_5\xi^3_6{\xi^1_6}^2\xi^1_1 - \xi^2_6{\xi^1_6}^2)/({\xi^1_6}^2(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6));$ $\mathbf{J^6_5} = (\xi^5_5\xi^2_6\xi^1_6 - \xi^4_5\xi^3_6\xi^1_6 + \xi^3_6\xi^2_6\xi^1_3 + {\xi^2_6}^2\xi^1_2 + \xi^2_6\xi^1_6\xi^1_1)/{\xi^1_6}^2;$
$\mathbf{J^6_6} = (\xi^5_6\xi^2_6 - \xi^4_6\xi^3_6 - \xi^3_6\xi^1_3 - \xi^2_6\xi^1_2 - \xi^1_6\xi^1_1)/\xi^1_6;$ and the parameters are subject to the condition

$$\xi^1_6(\xi^4_6\xi^2_6 + \xi^4_5\xi^1_6) \neq 0. \tag{8}$$

Now, equivalence by a suitable automorphism of the form

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & b^4_3 & 1 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & 0 & 1 & 0 \\ b^6_1 & b^6_2 & b^6_3 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

reduces to the case $\xi^1_1 = \xi^1_2 = \xi^1_3 = \xi^4_6 = \xi^5_2 = \xi^5_3 = \xi^5_6 = 0$. Then from (8), $\xi^4_5\xi^1_6 \neq 0$ and applying equivalence by $\Psi = \mathrm{diag}\left(\begin{pmatrix} 1 & 0 & 0 \\ \xi^2_6/\xi^1_6 & 0 & -\xi^4_5 c \\ \xi^3_6/\xi^1_6 & c & -\xi^5_5 c \end{pmatrix}, \begin{pmatrix} 0 & -\xi^4_5 c & 0 \\ c & -\xi^5_5 c & 0 \\ \xi^2_6 c/\xi^1_6 & c(-\xi^5_5\xi^2_6 + \xi^4_5\xi^3_6)/\xi^1_6 & \xi^4_5 c^2 \end{pmatrix}\right)$ where

$c=|\xi_5^4\xi_6^1|^{-\frac{1}{2}}$, we get reduced according to the sign of $\xi_5^4\xi_6^1$ to either

$$J_1=\begin{pmatrix}0&0&0&0&0&1\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&-1&0&0\\-1&0&0&0&0&0\end{pmatrix}\text{ or }J_1^-=\begin{pmatrix}0&0&0&0&0&-1\\0&0&1&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&-1&0&0\\1&0&0&0&0&0\end{pmatrix}. \tag{10}$$

Now $J_1^-$ is equivalent to $J_1$, hence any CS with $\xi_6^1\neq 0$ is equivalent to $J_1$.

### 3.2 Case $\xi_6^1=0, \xi_5^2\neq 0$.

$$J=\begin{pmatrix}-\xi_6^6&*&*&0&0&0\\ \boxed{\xi_1^2}&*&\boxed{\xi_3^2}&\boxed{\xi_4^2}&\boxed{\xi_5^2}&0\\ \boxed{\xi_1^3}&*&(\xi_6^6\xi_5^2-\xi_4^2\xi_3^2)/\xi_5^2&-{\xi_4^2}^2/\xi_5^2&-\xi_4^2&0\\ (-\xi_5^6\xi_3^2+\xi_3^6\xi_5^2+\xi_5^4\xi_1^2)/\xi_5^2&*&\boxed{\xi_3^4}&(-\xi_6^6\xi_5^2+\xi_5^4\xi_4^2)/\xi_5^2&\boxed{\xi_5^4}&\xi_5^2({\xi_6^6}^2+1)/(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2)\\ *&*&*&\xi_4^2(\xi_6^6+\xi_5^5)/\xi_5^2&\boxed{\xi_5^5}&-\xi_4^2({\xi_6^6}^2+1)/(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2)\\ *&*&\boxed{\xi_3^6}&(\xi_5^6\xi_4^2-\xi_1^3\xi_5^2-\xi_4^2\xi_1^2)/\xi_5^2&\boxed{\xi_5^6}&\boxed{\xi_6^6}\end{pmatrix} \tag{11}$$

where $\mathbf{J_3^1}=-(\xi_5^2({\xi_6^6}^2+1))/(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2)$; $\mathbf{J_2^2}=(-\xi_5^5\xi_5^2-\xi_5^4\xi_4^2+\xi_4^2\xi_3^2)/\xi_5^2$; $\mathbf{J_2^3}=(\xi_4^2(\xi_6^6\xi_5^2+\xi_5^5\xi_5^2+\xi_5^4\xi_4^2-\xi_4^2\xi_3^2))/{\xi_5^2}^2$; $\mathbf{J_2^4}=(-{\xi_6^6}^2\xi_5^6{\xi_5^2}^2+\xi_6^6\xi_5^4\xi_1^3{\xi_5^2}^2+\xi_6^6\xi_5^4\xi_5^2\xi_4^2\xi_1^2-\xi_5^6{\xi_5^2}^2-\xi_5^5\xi_5^4\xi_1^3{\xi_5^2}^2-\xi_5^5\xi_5^4\xi_5^2\xi_4^2\xi_1^2-{\xi_5^4}^2\xi_1^3\xi_5^2\xi_4^2-{\xi_5^4}^2{\xi_4^2}^2\xi_1^2+\xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2+\xi_3^4\xi_5^2{\xi_4^2}^2\xi_1^2)/({\xi_5^2}^2(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2))$; $\mathbf{J_1^5}=(\xi_6^6\xi_5^2\xi_1^2+\xi_5^6\xi_4^2\xi_3^2-\xi_3^6\xi_5^2\xi_4^2+\xi_5^5\xi_5^2\xi_1^2-\xi_1^3\xi_5^2\xi_3^2-\xi_4^2\xi_3^2\xi_1^2)/{\xi_5^2}^2$; $\mathbf{J_2^5}=({\xi_6^6}^2\xi_5^6{\xi_5^2}^2\xi_4^2+{\xi_6^6}^2{\xi_5^2}^2\xi_4^2\xi_1^2-\xi_6^6\xi_5^4\xi_1^3{\xi_5^2}^2\xi_4^2-\xi_6^6\xi_5^4\xi_5^2{\xi_4^2}^2\xi_1^2-\xi_6^6\xi_1^3{\xi_5^2}^2\xi_4^2\xi_3^2-\xi_6^6\xi_5^2{\xi_4^2}^2\xi_3^2\xi_1^2+\xi_5^6{\xi_5^2}^2\xi_4^2-{\xi_5^5}^2\xi_1^3{\xi_5^2}^3-{\xi_5^5}^2{\xi_5^2}^2\xi_4^2\xi_1^2-\xi_5^5\xi_5^4\xi_1^3{\xi_5^2}^2\xi_4^2-\xi_5^5\xi_5^4\xi_5^2{\xi_4^2}^2\xi_1^2+\xi_5^5\xi_1^3{\xi_5^2}^2\xi_4^2\xi_3^2+\xi_5^5\xi_5^2{\xi_4^2}^2\xi_3^2\xi_1^2+\xi_5^4\xi_1^3\xi_5^2{\xi_4^2}^2\xi_3^2+\xi_5^4{\xi_4^2}^3\xi_3^2\xi_1^2-\xi_3^4\xi_1^3{\xi_5^2}^2{\xi_4^2}^2-\xi_3^4\xi_5^2{\xi_4^2}^3\xi_1^2-\xi_1^3{\xi_5^2}^3)/({\xi_5^2}^3(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2))$; $\mathbf{J_3^5}=({\xi_6^6}^2{\xi_5^2}^2\xi_1^2-\xi_6^6\xi_1^3{\xi_5^2}^2\xi_3^2-\xi_6^6\xi_5^2\xi_4^2\xi_3^2\xi_1^2+\xi_5^5\xi_1^3{\xi_5^2}^2\xi_3^2+\xi_5^5\xi_5^2\xi_4^2\xi_3^2\xi_1^2+\xi_5^4\xi_1^3\xi_5^2\xi_4^2\xi_3^2+\xi_5^4{\xi_4^2}^2\xi_3^2\xi_1^2-\xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2-\xi_3^4\xi_5^2{\xi_4^2}^2\xi_1^2+{\xi_5^2}^2\xi_1^2)/({\xi_5^2}^2(\xi_1^3\xi_5^2+\xi_4^2\xi_1^2))$; $\mathbf{J_1^6}=({\xi_6^6}^2\xi_5^6{\xi_5^2}^2\xi_1^2-2\xi_6^6\xi_5^6\xi_1^3{\xi_5^2}^2\xi_3^2-2\xi_6^6\xi_5^6\xi_5^2\xi_4^2\xi_3^2\xi_1^2+2\xi_6^6\xi_3^6\xi_1^3{\xi_5^2}^3+2\xi_6^6\xi_3^6{\xi_5^2}^2\xi_4^2\xi_1^2+\xi_5^6{\xi_5^2}^2\xi_1^2+\xi_5^4{\xi_1^3}^2{\xi_5^2}^2\xi_3^2+2\xi_5^4\xi_1^3\xi_5^2\xi_4^2\xi_3^2\xi_1^2+\xi_5^4{\xi_4^2}^2\xi_3^2{\xi_1^2}^2-\xi_3^4{\xi_1^3}^2{\xi_5^2}^3-2\xi_3^4\xi_1^3{\xi_5^2}^2\xi_4^2\xi_1^2-\xi_3^4\xi_5^2{\xi_4^2}^2{\xi_1^2}^2)/({\xi_5^2}^3({\xi_6^6}^2+1))$; $\mathbf{J_2^6}=(-\xi_6^6\xi_5^6\xi_5^2-\xi_5^6\xi_5^5\xi_5^2-\xi_5^6\xi_5^4\xi_4^2+\xi_3^6\xi_5^2\xi_4^2+\xi_5^4\xi_1^3\xi_5^2+\xi_5^4\xi_4^2\xi_1^2)/{\xi_5^2}^2$; and the parameters are subject to the condition

$$\xi_5^2(\xi_5^2\xi_1^3+\xi_4^2\xi_1^2)\neq 0. \tag{12}$$

Now, equivalence by a suitable automorphism of the form (9) reduces to the case $\xi_1^2=\xi_3^2=\xi_5^4=\xi_5^5=\xi_5^6=\xi_3^4=\xi_3^6=0$. Applying then equivalence by $\Psi=\operatorname{diag}\left(\begin{pmatrix}-\xi_6^6/\xi_5^2&-\xi_6^6\xi_4^2/(\xi_1^3\xi_5^2)&-1/\xi_1^3\\0&1&0\\\xi_1^3/\xi_5^2&0&0\end{pmatrix},\begin{pmatrix}-\xi_6^6/\xi_5^2&0&1/\xi_1^3\\\xi_6^6\xi_4^2/{\xi_5^2}^2&1/\xi_5^2&0\\-\xi_1^3/\xi_5^2&0&0\end{pmatrix}\right)$, $J2=\Psi^{-1}J\Psi$ is

$$J2=\begin{pmatrix}0&0&-\xi_5^2/\xi_1^3&0&-\xi_4^2/\xi_1^3&-{\xi_4^2}^2/{\xi_1^3}^2\\0&0&0&0&1&\xi_4^2/\xi_1^3\\\xi_1^3/\xi_5^2&\xi_4^2/\xi_5^2&0&0&0&0\\0&0&0&0&0&\xi_5^2/\xi_1^3\\0&-1&0&\xi_4^2/\xi_5^2&0&0\\0&0&0&-\xi_1^3/\xi_5^2&0&0\end{pmatrix}.$$

Suppose first that $\xi_4^2\neq 0$. Then this $J2$ is a CS belonging in the case 3.1. Suppose now that $\xi_4^2=0$. Applying equivalence by the automorphism $\Lambda=\operatorname{diag}(1,\xi_1^3/\xi_5^2,\xi_1^3/\xi_5^2,\xi_1^3/\xi_5^2,\xi_1^3/\xi_5^2,{\xi_1^3}^2/{\xi_5^2}^2)$, $\Lambda^{-1}J2\Lambda$ is the matrix

$$J_2=\begin{pmatrix}0&0&-1&0&0&0\\0&0&0&0&1&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&-1&0&0&0&0\\0&0&0&-1&0&0\end{pmatrix} \tag{13}$$

Hence, from the result of the case 3.1, any CS with $\xi_6^1=0,\xi_5^2\neq 0$ is equivalent to either $J_1$ in (10) or $J_2$ in (13). Since $J_2$ is equivalent to $J_1$ by the automorphism $M=\operatorname{diag}\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),-1,0,0,1,\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\right)$ *i.e.* $M^{-1}J_2M=J_1$, we get that any CS with $\xi_6^1=0,\xi_5^2\neq 0$ is equivalent to $J_1$.

### 3.3 Case $\xi^1_6 = 0, \xi^2_5 = 0$.

$$J = \begin{pmatrix} \boxed{\xi^1_1} & \boxed{\xi^1_2} & 0 & 0 & 0 & 0 \\ -\frac{{\xi^1_1}^2+1}{\xi^1_2} & -\xi^1_1 & 0 & 0 & 0 & 0 \\ \boxed{\xi^3_1} & \frac{\xi^1_2(\xi^4_1\xi^3_4+\xi^3_3\xi^3_1+\xi^3_1\xi^1_1)}{{\xi^1_1}^2+1} & \boxed{\xi^3_3} & \boxed{\xi^3_4} & 0 & 0 \\ \boxed{\xi^4_1} & \frac{\xi^1_2(-\xi^4_1\xi^3_4\xi^3_3+\xi^4_1\xi^3_4\xi^1_1-{\xi^3_3}^2\xi^3_1-\xi^3_1)}{\xi^3_4({\xi^1_1}^2+1)} & -\frac{{\xi^3_3}^2+1}{\xi^3_4} & -\xi^3_3 & 0 & 0 \\ \frac{\xi^6_4\xi^4_1\xi^1_2+\xi^6_3\xi^3_1\xi^1_2-\xi^6_2{\xi^1_1}^2-\xi^6_2}{{\xi^1_1}^2+1} & * & * & \frac{\xi^1_2(-\xi^6_4\xi^3_3-\xi^6_4\xi^1_1+\xi^6_3\xi^3_4)}{{\xi^1_1}^2+1} & \xi^1_1 & \xi^1_2 \\ \boxed{\xi^6_1} & \boxed{\xi^6_2} & \boxed{\xi^6_3} & \boxed{\xi^6_4} & -\frac{{\xi^1_1}^2+1}{\xi^1_2} & -\xi^1_1 \end{pmatrix} \tag{14}$$

where $\mathbf{J^5_2} = (\xi^1_2(-\xi^6_4\xi^4_1\xi^3_4\xi^3_3\xi^1_2 + \xi^6_4\xi^4_1\xi^3_4\xi^1_2\xi^1_1 - \xi^6_4{\xi^3_3}^2\xi^3_1\xi^1_2 - \xi^6_4\xi^3_1\xi^1_2 + \xi^6_3\xi^4_1{\xi^3_4}^2\xi^1_2 + \xi^6_3\xi^3_4\xi^3_3\xi^3_1\xi^1_2 + \xi^6_3\xi^3_4\xi^3_1\xi^1_2\xi^1_1 - 2\xi^6_2\xi^3_4{\xi^1_1}^3 - 2\xi^6_2\xi^3_4\xi^1_1 + \xi^6_1\xi^3_4\xi^1_2{\xi^1_1}^2 + \xi^6_1\xi^3_4\xi^1_2))/(\xi^3_4({\xi^1_1}^2+1)^2)$; $\mathbf{J^5_3} = (\xi^1_2(-\xi^6_4{\xi^3_3}^2 - \xi^6_4 + \xi^6_3\xi^3_4\xi^3_3 - \xi^6_3\xi^3_4\xi^1_1))/(\xi^3_4({\xi^1_1}^2+1))$; and the parameters are subject to the condition

$$\xi^1_2\xi^3_4 \neq 0. \tag{15}$$

Now, equivalence by a suitable automorphism of the form (9) reduces to the case $\xi^3_1 = \xi^3_3 = \xi^4_1 = \xi^6_1 = \xi^6_2 = \xi^6_3 = \xi^6_4 = 0$. Applying then equivalence by the automorphism $\Psi = \mathrm{diag}\left(\begin{pmatrix} \xi^1_2 & 0 & 0 \\ -\xi^1_1 & 1 & 0 \\ 0 & 0 & \xi^3_4\xi^1_2 \end{pmatrix}, \begin{pmatrix} \xi^1_2 & 0 & 0 \\ 0 & \xi^3_4{\xi^1_2}^2 & 0 \\ 0 & -\xi^3_4\xi^1_2\xi^1_1 & \xi^3_4\xi^1_2 \end{pmatrix}\right)$, we get into the case of a CS $J$ where all parameters but $\xi^1_2 = \xi^3_4 = 1$ vanish and $\xi^1_2 = \xi^3_4 = 1$, that is $J = \mathrm{diag}\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$, which is equivalent to the matrix $J_1$ in (10). Now, with the automorphism $\mathrm{diag}(1,-1,1,-1,1,-1)$, $J$ is equivalent to its opposite

$$J_0 = \mathrm{diag}\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right) . \tag{16}$$

Hence, any CS with $\xi^1_6 = 0, \xi^2_5 = 0$ is equivalent to $J_0$ in (16). Commutation relations of $\mathfrak{m}$ for $J_0 : [\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = -\tilde{x}_5$.

### 3.4 Conclusions.

Any $J \in \mathfrak{X}_{6,3}$ is equivalent to $J_0$ defined by (16). Hence $\mathfrak{X}_{6,3}$ is comprised of the single Aut $\mathcal{G}_{6,3}$ orbit of $J_0$. Now Aut $\mathcal{G}_{6,3}$ consists of the matrices

$$\Phi = \begin{pmatrix} b^1_1 & b^1_2 & b^1_3 & 0 & 0 & 0 \\ b^2_1 & b^2_2 & b^2_3 & 0 & 0 & 0 \\ b^3_1 & b^3_2 & b^3_3 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & b^4_3 & b^2_2b^1_1 - b^2_1b^1_2 & b^2_3b^1_1 - b^2_1b^1_3 & b^2_3b^1_2 - b^2_2b^1_3 \\ b^5_1 & b^5_2 & b^5_3 & b^3_2b^1_1 - b^3_1b^1_2 & b^3_3b^1_1 - b^3_1b^1_3 & b^3_3b^1_2 - b^3_2b^1_3 \\ b^6_1 & b^6_2 & b^6_3 & b^3_2b^2_1 - b^3_1b^2_2 & b^3_3b^2_1 - b^3_1b^2_3 & b^3_3b^2_2 - b^3_2b^2_3 \end{pmatrix}$$

where the $b^i_j$'s are arbitrary reals with the condition $\det\Phi \neq 0$, and the stabilizer of $J_0$ is 6-dimensional. Hence $\mathfrak{X}_{6,3}$ is a submanifold of dimension 12 of $\mathbb{R}^{36}$ ([1], Chap. 3, par. 1, Prop. 14). We also remark that $\mathfrak{X}_{6,3}$ is the zero set of a polynomial map $F : \mathbb{R}^{36} \to \mathbb{R}^{81} \times \mathbb{R}^{36}$; however this map is not a subimmersion, that is its rank is not locally constant.

### 3.5

$$X_1 = \frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^3} - y^1\frac{\partial}{\partial y^2} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - x^2\frac{\partial}{\partial y^3} .$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J_0$ (16). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \; ; \; \tilde{X}^-_j f = 0 \; \forall j = 1,3,5\}$. One has

$$\tilde{X}^-_1 = 2\frac{\partial}{\partial \overline{z^1}} - 2\,x^2\frac{\partial}{\partial \overline{z^3}} - y^1\frac{\partial}{\partial y^2} \; ; \; \tilde{X}^-_3 = 2\frac{\partial}{\partial \overline{z^2}} \; ; \; \tilde{X}^-_5 = 2\frac{\partial}{\partial \overline{z^3}}$$

where $z^j = x^j + iy^j$ $(1 \leqslant j \leqslant 3)$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $z^2$ and $z^3$ and satisfies

$$2\,\frac{\partial f}{\partial \overline{z^1}} = \frac{z^1 - \overline{z^1}}{2}\,\frac{\partial f}{\partial z^2}\,.$$

Hence the 3 functions $w^1 = z^1, w^2 = z^2 + \frac{|z^1|^2}{4} - \frac{(\overline{z^1})^2}{8}, w^3 = z^3$ are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (w^1, w^2, w^3)$. $F$ is a biholomorphic bijection, hence a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$. Computations yield :

$$\begin{aligned}
w_{ax}^1 &= w_a^1 + w_x^1 \\
w_{ax}^2 &= w_a^2 + w_x^2 - \frac{w_a^1 - \overline{w_a^1}}{4}\, w_x^1 \\
w_{ax}^3 &= w_a^3 + w_x^3 + \frac{1}{2}(w_a^2 - \overline{w_a^2} + \frac{(\overline{w_a^1})^2 - (w_a^1)^2}{4} - |w_a^1|^2)\, w_x^1.
\end{aligned}$$

# 4 Lie Algebra $\mathcal{G}_{6,7}$ (isomorphic to $M6$).

Commutation relations for $\mathcal{G}_{6,7}$ : $[x_1, x_2] = x_4$; $[x_1, x_3] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = -x_6$.

$$J = \begin{pmatrix}
\boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\
-\frac{{\xi_1^1}^2+1}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0 \\
* & \boxed{\xi_2^3} & c & \boxed{\xi_4^3} & 0 & 0 \\
* & \boxed{\xi_2^4} & -\frac{c^2+1}{\xi_4^3} & -c & 0 & 0 \\
* & * & * & \boxed{\xi_4^5} & \boxed{\xi_5^5} & -\frac{\xi_4^3\xi_2^1}{\xi_4^3-\xi_2^1} \\
\boxed{\xi_1^6} & \boxed{\xi_2^6} & * & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2+1)(\xi_4^3-\xi_2^1)}{\xi_4^3\xi_2^1} & -\xi_5^5
\end{pmatrix} \tag{17}$$

where $\mathbf{J_1^3} = (\xi_5^5\xi_4^3\xi_2^3 - \xi_5^5\xi_2^3\xi_2^1 - \xi_2^4\xi_4^3\xi_2^1 - \xi_4^3\xi_2^3\xi_1^1 + \xi_2^3\xi_2^1\xi_1^1)/{\xi_2^1}^2$; $\mathbf{c} = \mathbf{J_3^3} = (-\xi_5^5\xi_4^3 + \xi_5^5\xi_2^1 + \xi_4^3\xi_1^1)/\xi_2^1$; $\mathbf{J_1^4} = ({\xi_5^5}^2{\xi_4^3}^2\xi_2^3 - 2{\xi_5^5}^2\xi_4^3\xi_2^3\xi_2^1 + {\xi_5^5}^2\xi_2^3{\xi_2^1}^2 - \xi_5^5\xi_2^4{\xi_4^3}^2\xi_2^1 + \xi_5^5\xi_2^4\xi_4^3{\xi_2^1}^2 - 2\xi_5^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2\xi_5^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 + \xi_2^4{\xi_4^3}^2\xi_2^1\xi_1^1 + \xi_2^4\xi_4^3{\xi_2^1}^2\xi_1^1 + {\xi_4^3}^2\xi_2^3{\xi_1^1}^2 + \xi_2^3{\xi_2^1}^2)/(\xi_4^3{\xi_2^1}^3)$; $\mathbf{J_1^5} = (-\xi_4^6{\xi_5^5}^2{\xi_4^3}^2\xi_2^3\xi_2^1 + \xi_4^6{\xi_5^5}^2\xi_4^3\xi_2^3{\xi_2^1}^2 + \xi_4^6\xi_5^5\xi_2^4{\xi_4^3}^2{\xi_2^1}^2 + 2\xi_4^6\xi_5^5{\xi_4^3}^2\xi_2^3\xi_2^1\xi_1^1 - 2\xi_4^6\xi_5^5\xi_4^3\xi_2^3{\xi_2^1}^2\xi_1^1 - \xi_4^6\xi_2^4{\xi_4^3}^2{\xi_2^1}^2\xi_1^1 - \xi_4^6{\xi_4^3}^2\xi_2^3\xi_2^1{\xi_1^1}^2 - \xi_4^6\xi_4^3\xi_2^3{\xi_2^1}^2 + \xi_2^6{\xi_4^3}^2{\xi_2^1}^2{\xi_1^1}^2 + \xi_2^6{\xi_4^3}^2{\xi_2^1}^2 + \xi_1^6\xi_5^5{\xi_4^3}^2{\xi_2^1}^3 - \xi_1^6{\xi_4^3}^2{\xi_2^1}^3\xi_1^1 + {\xi_5^5}^3\xi_4^5{\xi_4^3}^2\xi_2^3 - 2{\xi_5^5}^3\xi_4^5\xi_4^3\xi_2^3\xi_2^1 + {\xi_5^5}^3\xi_4^5\xi_2^3{\xi_2^1}^2 - {\xi_5^5}^2\xi_4^5\xi_2^4{\xi_4^3}^2\xi_2^1 + {\xi_5^5}^2\xi_4^5\xi_2^4\xi_4^3{\xi_2^1}^2 - {\xi_5^5}^2\xi_4^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2{\xi_5^5}^2\xi_4^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 - {\xi_5^5}^2\xi_4^5\xi_2^3{\xi_2^1}^2\xi_1^1 + \xi_5^5\xi_4^5{\xi_4^3}^2\xi_2^3 - 2\xi_5^5\xi_4^5\xi_4^3\xi_2^3\xi_2^1 + \xi_5^5\xi_4^5\xi_2^3{\xi_2^1}^2 - \xi_4^5\xi_2^4{\xi_4^3}^2\xi_2^1 + \xi_4^5\xi_2^4\xi_4^3{\xi_2^1}^2 - \xi_4^5{\xi_4^3}^2\xi_2^3\xi_1^1 + 2\xi_4^5\xi_4^3\xi_2^3\xi_2^1\xi_1^1 - \xi_4^5\xi_2^3{\xi_2^1}^2\xi_1^1)/(\xi_4^3{\xi_2^1}^2({\xi_5^5}^2 + 1)(\xi_4^3 - \xi_2^1))$; $\mathbf{J_2^5} = (\xi_4^6\xi_5^5{\xi_4^3}^2\xi_2^3 - 2\xi_4^6\xi_5^5\xi_4^3\xi_2^3\xi_2^1 - \xi_4^6\xi_2^4{\xi_4^3}^2\xi_2^1 - \xi_4^6{\xi_4^3}^2\xi_2^3\xi_1^1 + \xi_2^6\xi_5^5{\xi_4^3}^2\xi_2^1 + \xi_2^6{\xi_4^3}^2\xi_2^1\xi_1^1 - \xi_1^6{\xi_4^3}^2{\xi_2^1}^2 + {\xi_5^5}^2\xi_4^5\xi_4^3\xi_2^3 - {\xi_5^5}^2\xi_4^5\xi_2^3\xi_2^1 + \xi_4^5\xi_4^3\xi_2^3 - \xi_4^5\xi_2^3\xi_2^1)/(\xi_4^3({\xi_5^5}^2 + 1)(\xi_4^3 - \xi_2^1))$; $\mathbf{J_3^5} = (\xi_4^6{\xi_2^1}^2 - \xi_5^5\xi_4^5\xi_4^3 + \xi_5^5\xi_4^5\xi_2^1 + \xi_4^5\xi_4^3\xi_1^1 - \xi_4^5\xi_2^1\xi_1^1)/(\xi_2^1(\xi_4^3 - \xi_2^1))$; $\mathbf{J_3^6} = (-\xi_4^6\xi_5^5{\xi_4^3}^2 + 2\xi_4^6\xi_5^5\xi_4^3\xi_2^1 + \xi_4^6{\xi_4^3}^2\xi_1^1 - {\xi_5^5}^2\xi_4^5\xi_4^3 + {\xi_5^5}^2\xi_4^5\xi_2^1 - \xi_4^5\xi_4^3 + \xi_4^5\xi_2^1)/({\xi_4^3}^2\xi_2^1)$;
and the parameters are subject to the condition

$$\xi_2^1\xi_4^3(\xi_2^1 - \xi_4^3) \neq 0. \tag{18}$$

Now the automorphism group of $\mathcal{G}_{6,7}$ is comprised of the matrices

$$\Phi = \begin{pmatrix}
b_1^1 & 0 & 0 & 0 & 0 & 0 \\
b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\
b_1^3 & b_2^3 & {b_1^1}^2 & 0 & 0 & 0 \\
b_1^4 & b_2^4 & b_3^4 & b_2^2 b_1^1 & 0 & 0 \\
b_1^5 & b_2^5 & b_3^5 & b_2^3 b_1^1 & {b_1^1}^3 & 0 \\
b_1^6 & b_2^6 & b_3^6 & b_2^4 b_1^1 - b_2^3 b_1^2 + b_1^3 b_2^2 & b_1^1(b_3^4 - b_1^2 b_1^1) & b_2^2 {b_1^1}^2
\end{pmatrix}$$

where $b_2^2 b_1^1 \neq 0$. Taking suitable values for the $b_j^i$'s, equivalence by $\Phi$ leads to the case where $\xi_1^1 = \xi_2^3 = \xi_2^4 = \xi_4^5 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_4^6 = 0$ and $\xi_2^1 = 1, \xi_4^3 = \alpha$, where $\alpha \neq 0, 1$:

$$J_\alpha = \operatorname{diag}\left( \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & \alpha \\ -1/\alpha & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -\alpha/(\alpha-1) \\ (\alpha-1)/\alpha & \end{smallmatrix}\right) \right) \quad (\alpha \neq 0, 1) \tag{19}$$

Hence, any CS is equivalent to $J_\alpha$ in (19). It is easily seen that the $J_\alpha$'s corresponding to different values of $\alpha$ are not equivalent. Commutation relations of $\mathfrak{m}$ for $J_\alpha : [\tilde{x}_1, \tilde{x}_3] = \tilde{x}_5 \, ; \, [\tilde{x}_1, \tilde{x}_4] = (1-\alpha)\tilde{x}_6 \, ; \, [\tilde{x}_2, \tilde{x}_3] = \frac{1-\alpha}{\alpha} \, \tilde{x}_6 \, ; \, [\tilde{x}_2, \tilde{x}_4] = -\alpha \tilde{x}_5$.

From (17), $\mathfrak{X}_{6,7}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J_\alpha$'s in (19).

### 4.1

$$X_1 = \frac{\partial}{\partial x^1} - y^1 \frac{\partial}{\partial y^2} - x^2 \frac{\partial}{\partial x^3} - y^2 \frac{\partial}{\partial y^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J_\alpha$ in (19). Then $H_{\mathbb{C}}(G) = \{ f \in C^\infty(G_0) \, ; \, \tilde{X}_j^- \, f = 0 \; \forall j = 1,3,5 \}$. One has

$$\tilde{X}_1^- = 2 \, \frac{\partial}{\partial \overline{w^1}} - y^1 \, \frac{\partial}{\partial y^2} - x^2 \, \frac{\partial}{\partial x^3} - (y^2 + i x^2) \, \frac{\partial}{\partial y^3} \quad , \quad \tilde{X}_3^- = 2 \, \frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}_5^- = 2 \, \frac{\partial}{\partial \overline{w^3}} \quad ,$$

where $w^1 = x^1 - i y^1 \quad , \quad w^2 = x^2 - i \alpha y^2 \quad , \quad w^3 = x^2 + i \frac{\alpha}{\alpha - 1} \, y^3$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies

$$2 \, \frac{\partial f}{\partial \overline{w^1}} - \alpha \, \frac{w^1 - \overline{w^1}}{2} \, \frac{\partial f}{\partial w^2} + \frac{w^2}{\alpha - 1} \, \frac{\partial f}{\partial w^3} = 0.$$

The 3 functions

$$\begin{aligned} \varphi^1 &= w^1 \quad , \quad \varphi^2 = w^2 + \alpha \left( -\frac{(\overline{w^1})^2}{8} + \frac{|w^1|^2}{4} \right) \quad , \\ \varphi^3 &= w^3 + \frac{\alpha}{8(1-\alpha)} \, (\overline{w^1})^2 \left( \frac{w^1}{2} - \frac{\overline{w^1}}{3} \right) + \frac{\overline{w^1} w^2}{2(1-\alpha)} \end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a biholomorphic bijection, hence a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, $a = [w_a^1, w_a^2, w_a^3]$, $x = [w_x^1, w_x^2, w_x^3]$, $a\,x = [w_{ax}^1, w_{ax}^2, w_{ax}^3]$, $a = [\varphi_a^1, \varphi_a^2, \varphi_a^3]$, $x = [\varphi_x^1, \varphi_x^2, \varphi_x^3]$, $a\,x = [\varphi_{ax}^1, \varphi_{ax}^2, \varphi_{ax}^3]$. Computations yield :

$$\begin{aligned} w_{ax}^1 &= w_a^1 + w_x^1 \quad , \quad w_{ax}^2 = w_a^2 + w_x^2 + i \alpha \beta^1 x^1 \quad , \\ w_{ax}^3 &= w_a^3 + w_x^3 - \alpha^2 x^1 + i \frac{\alpha}{\alpha - 1} \left( -\beta^1 x^1 + \alpha^2 y^1 + \frac{1}{2} \, \beta^1 (x^1)^2 \right). \end{aligned}$$

We then get

$$\varphi_{ax}^1 = \varphi_a^1 + \varphi_x^1 \quad , \quad \varphi_{ax}^2 = \varphi_a^2 + \varphi_x^2 + \frac{\alpha}{4} \left( 2\overline{\varphi_a^1} - \varphi_a^1 \right) \varphi_x^1 \quad , \quad \varphi_{ax}^3 = \varphi_a^3 + \varphi_x^3 + \chi(a, x)$$

where

$$\begin{aligned} \chi(a,x) = \varphi_x^1 \left( -\frac{1}{2} \, \varphi_a^2 + \frac{\alpha}{2(1-\alpha)} \, \overline{\varphi_a^2} + \frac{\alpha(2+\alpha)}{16(1-\alpha)} \, (\overline{\varphi_a^1})^2 + \frac{\alpha^2}{16(1-\alpha)} \, (\varphi_a^1)^2 - \frac{\alpha^2}{4(1-\alpha)} \, |\varphi_a^1|^2 \right) \\ + \frac{\alpha}{16(1-\alpha)} \left( \varphi_a^1 - \overline{\varphi_a^1} \right) (\varphi_x^1)^2 + \frac{1}{2(1-\alpha)} \, \overline{\varphi_a^1} \, \varphi_x^2. \end{aligned}$$

# 5 Lie Algebra $\mathcal{G}_{6,4}$ (isomorphic to $M7$).

Commutation relations for $\mathcal{G}_{6,4}$ : $[x_1, x_2] = x_4$; $[x_1, x_3] = x_6$; $[x_2, x_4] = x_5$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\ -\frac{({\xi_1^1}^2+1)}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0 \\ * & \boxed{\xi_2^3} & b & \boxed{\xi_4^3} & 0 & 0 \\ * & \boxed{\xi_2^4} & -\frac{b^2+1}{\xi_4^3} & -b & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{c} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & c & -\xi_5^5 \end{pmatrix} \tag{20}$$

where $\mathbf{J_1^3} = (-\xi_5^5\xi_2^4\xi_4^3 + 2\xi_5^5\xi_2^3\xi_1^1 - \xi_2^4\xi_4^3\xi_1^1 + \xi_2^3{\xi_1^1}^2 - \xi_2^3)/(\xi_2^1(\xi_5^5+\xi_1^1))$; $\mathbf{b} = \mathbf{J_3^3} = (-\xi_5^5\xi_1^1+1)/(\xi_5^5+\xi_1^1)$; $\mathbf{J_1^4} = ({\xi_5^5}^2\xi_2^3{\xi_1^1}^2 + {\xi_5^5}^2\xi_2^3 + \xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^2 + \xi_5^5\xi_2^4\xi_4^3 + \xi_2^4\xi_4^3{\xi_1^1}^3 + \xi_2^4\xi_4^3\xi_1^1 + \xi_2^3{\xi_1^1}^2 + \xi_2^3)/(\xi_4^3\xi_2^1(\xi_5^5+\xi_1^1)^2)$; $\mathbf{J_1^5} = (\xi_4^6{\xi_5^5}^2\xi_2^3{\xi_1^1}^4 + 2\xi_4^6{\xi_5^5}^2\xi_2^3{\xi_1^1}^2 + \xi_4^6{\xi_5^5}^2\xi_2^3 + \xi_4^6\xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^4 + 2\xi_4^6\xi_5^5\xi_2^4\xi_4^3{\xi_1^1}^2 + \xi_4^6\xi_5^5\xi_2^4\xi_4^3 + \xi_4^6\xi_2^4\xi_4^3{\xi_1^1}^5 + 2\xi_4^6\xi_2^4\xi_4^3{\xi_1^1}^3 + \xi_4^6\xi_2^4\xi_4^3\xi_1^1 + \xi_4^6\xi_2^3{\xi_1^1}^4 + 2\xi_4^6\xi_2^3{\xi_1^1}^2 + \xi_4^6\xi_2^3 - \xi_3^6{\xi_5^5}^2\xi_2^4{\xi_4^3}^2{\xi_1^1}^2 - \xi_3^6{\xi_5^5}^2\xi_2^4{\xi_4^3}^2 + 2\xi_3^6{\xi_5^5}^2\xi_4^3\xi_2^3{\xi_1^1}^3 + 2\xi_3^6{\xi_5^5}^2\xi_4^3\xi_2^3\xi_1^1 - 2\xi_3^6\xi_5^5\xi_2^4{\xi_4^3}^2{\xi_1^1}^3 - 2\xi_3^6\xi_5^5\xi_2^4{\xi_4^3}^2\xi_1^1 + 3\xi_3^6\xi_5^5\xi_4^3\xi_2^3{\xi_1^1}^4 + 2\xi_3^6\xi_5^5\xi_4^3\xi_2^3{\xi_1^1}^2 - \xi_3^6\xi_5^5\xi_4^3\xi_2^3 - \xi_3^6\xi_2^4{\xi_4^3}^2{\xi_1^1}^4 - \xi_3^6\xi_2^4{\xi_4^3}^2{\xi_1^1}^2 + \xi_3^6\xi_4^3\xi_2^3{\xi_1^1}^5 - \xi_3^6\xi_4^3\xi_2^3\xi_1^1 - \xi_2^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^4 - 2\xi_2^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^2 - \xi_2^6{\xi_5^5}^2\xi_4^3 - 2\xi_2^6\xi_5^5\xi_4^3{\xi_1^1}^5 - 4\xi_2^6\xi_5^5\xi_4^3{\xi_1^1}^3 - 2\xi_2^6\xi_5^5\xi_4^3\xi_1^1 - \xi_2^6\xi_4^3{\xi_1^1}^6 - 2\xi_2^6\xi_4^3{\xi_1^1}^4 - \xi_2^6\xi_4^3{\xi_1^1}^2 - \xi_1^6{\xi_5^5}^3\xi_4^3\xi_2^1{\xi_1^1}^2 - \xi_1^6{\xi_5^5}^3\xi_4^3\xi_2^1 - \xi_1^6{\xi_5^5}^2\xi_4^3\xi_2^1{\xi_1^1}^3 - \xi_1^6{\xi_5^5}^2\xi_4^3\xi_2^1\xi_1^1 + \xi_1^6\xi_5^5\xi_4^3\xi_2^1{\xi_1^1}^4 + \xi_1^6\xi_5^5\xi_4^3\xi_2^1{\xi_1^1}^2 + \xi_1^6\xi_4^3\xi_2^1{\xi_1^1}^5 + \xi_1^6\xi_4^3\xi_2^1{\xi_1^1}^3)/({\xi_4^3}^2{\xi_2^1}^2(\xi_5^5+\xi_1^1)^3)$; $\mathbf{J_2^5} = (\xi_4^6\xi_2^4{\xi_1^1}^2 + \xi_4^6\xi_2^4 + \xi_3^6\xi_2^3{\xi_1^1}^2 + \xi_3^6\xi_2^3 - \xi_2^6\xi_5^5{\xi_1^1}^2 - \xi_2^6\xi_5^5 - \xi_2^6{\xi_1^1}^3 - \xi_2^6\xi_1^1 + \xi_1^6\xi_2^1{\xi_1^1}^2 + \xi_1^6\xi_2^1)/(\xi_4^3\xi_2^1(\xi_5^5+\xi_1^1))$; $\mathbf{J_3^5} = (-\xi_4^6{\xi_5^5}^2{\xi_1^1}^4 - 2\xi_4^6{\xi_5^5}^2{\xi_1^1}^2 - \xi_4^6{\xi_5^5}^2 - \xi_4^6{\xi_1^1}^4 - 2\xi_4^6{\xi_1^1}^2 - \xi_4^6 - \xi_3^6{\xi_5^5}^3\xi_4^3{\xi_1^1}^2 - \xi_3^6{\xi_5^5}^3\xi_4^3 - 3\xi_3^6{\xi_5^5}^2\xi_4^3{\xi_1^1}^3 - 3\xi_3^6{\xi_5^5}^2\xi_4^3\xi_1^1 - 2\xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^4 - \xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^2 + \xi_3^6\xi_5^5\xi_4^3 + \xi_3^6\xi_4^3{\xi_1^1}^3 + \xi_3^6\xi_4^3\xi_1^1)/({\xi_4^3}^2\xi_2^1(\xi_5^5+\xi_1^1)^3)$; $\mathbf{J_4^5} = (-\xi_4^6{\xi_5^5}^2{\xi_1^1}^2 - \xi_4^6{\xi_5^5}^2 - \xi_4^6{\xi_1^1}^2 - \xi_4^6 + \xi_3^6\xi_5^5\xi_4^3{\xi_1^1}^2 + \xi_3^6\xi_5^5\xi_4^3 + \xi_3^6\xi_4^3{\xi_1^1}^3 + \xi_3^6\xi_4^3\xi_1^1)/(\xi_4^3\xi_2^1(\xi_5^5+\xi_1^1)^2)$; and the parameters are subject to the condition

$$\xi_2^1\xi_4^3(\xi_1^1 + \xi_5^5) \neq 0. \tag{21}$$

Now the automorphism group of $\mathcal{G}_{6,4}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2 b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4 b_2^2 & {b_2^2}^2 b_1^1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_2^3 b_1^1 & 0 & b_3^3 b_1^1 \end{pmatrix}$$

where $b_3^3 b_2^2 b_1^1 \neq 0$. Taking suitable values for the $b_j^i$'s, equivalence by $\Phi$ leads to the case where $\xi_2^3 = \xi_2^4 = \xi_2^6 = \xi_1^6 = \xi_3^6 = \xi_4^6 = 0$, $\xi_2^1 = \xi_4^3 = 1$ and $\xi_1^1 = \alpha, \xi_5^5 = \beta, \alpha \neq -\beta$ :

$$J_{\alpha,\beta} = \operatorname{diag}\left(\begin{pmatrix} \alpha & 1 \\ -(\alpha^2+1) & -\alpha \end{pmatrix}, \begin{pmatrix} \frac{-\alpha\beta+1}{\alpha+\beta} & 1 \\ -\frac{(\alpha^2+1)(\beta^2+1)}{(\alpha+\beta)^2} & \frac{\alpha\beta-1}{\alpha+\beta} \end{pmatrix}, \begin{pmatrix} \beta & \frac{(\alpha^2+1)(\beta^2+1)}{\alpha+\beta} \\ -\frac{\alpha+\beta}{\alpha^2+1} & -\beta \end{pmatrix}\right) \quad (\alpha \neq -\beta) \tag{22}$$

Hence, any CS is equivalent to $J_{\alpha,\beta}$ in (22). The $J_{\alpha,\beta}$'s corresponding to different couples $(\alpha, \beta)$ are not equivalent. Commutation relations of $\mathfrak{m}$ for $J_{\alpha,\beta}$ : $[\tilde{x}_1, \tilde{x}_3] = -\frac{(\beta^2+1)(\alpha^2+1)^2}{(\alpha+\beta)^2}\,\tilde{x}_5 - \frac{\beta(1+\alpha^2)}{\alpha+\beta}\,\tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_4] = \frac{\alpha\beta-1}{\alpha+\beta}\,(1+\alpha^2)\tilde{x}_5 - \alpha\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = -\frac{\alpha}{(\alpha+\beta)^2}\,(1+\alpha^2)(1+\beta^2)\tilde{x}_5 + \frac{\alpha\beta-1}{\alpha+\beta}\,\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = \beta\,\frac{\alpha^2+1}{\alpha+\beta}\,\tilde{x}_5 - \tilde{x}_6$.

From (20), $\mathfrak{X}_{6,4}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J_{\alpha,\beta}$'s in (22).

## 5.1

$$X_1 = \frac{\partial}{\partial x^1} + \frac{1}{2}(y^1)^2\,\frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial y^3} - y^1\frac{\partial}{\partial y^2} \quad , \quad X_2 = \frac{\partial}{\partial y^1} + y^2\frac{\partial}{\partial x^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined

by $J_{\alpha,\beta}$ (22). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$\begin{aligned}
\tilde{X}_1^- &= (1+i\alpha)\left[2\frac{\partial}{\partial \overline{w^1}} + \left(\frac{1}{2}(y^1)^2 + iy^2(1-i\alpha)\right)\frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial x^3} - y^1\frac{\partial}{\partial y^2}\right] \quad , \\
\tilde{X}_3^- &= 2\frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}_5^- = 2\frac{\partial}{\partial \overline{w^3}} \quad ,
\end{aligned}$$

where

$$\begin{aligned}
w^1 &= x^1 - i(\alpha x^1 + y^1) \\
w^2 &= x^2 + \frac{(1-\alpha\beta)(\alpha+\beta)}{(1+\alpha^2)(1+\beta^2)}y^2 - i\frac{(\alpha+\beta)^2}{(1+\alpha^2)(1+\beta^2)}y^2 \\
w^3 &= x^3 + \frac{\beta(\alpha^2+1)}{\alpha+\beta}y^3 - i\frac{\alpha^2+1}{\alpha+\beta}y^3.
\end{aligned}$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$2\frac{\partial f}{\partial \overline{w^1}} + \left(\frac{1}{4}((\alpha-i)w^1 + (\alpha+i)\overline{w^1})^2 - \frac{(\alpha^2+1)(\beta-i)}{\alpha+\beta}w^2\right)\frac{\partial f}{\partial w^3} + \frac{A}{2}((\alpha-i)w^1 + (\alpha+i)\overline{w^1})\frac{\partial f}{\partial w^2} = 0$$

where

$$A = \frac{(\alpha+\beta)(1-\alpha\beta - i(\alpha+\beta))}{(1+\alpha^2)(1+\beta^2)}.$$

The 3 functions

$$\varphi^1 = w^1 \quad , \quad \varphi^2 = w^2 - \frac{A}{4}\left(\frac{\alpha+i}{2}(\overline{w^1})^2 + (\alpha-i)\,|w^1|^2\right) \quad ,$$

$$\begin{aligned}
\varphi^3 = w^3 - \frac{1}{16}(\alpha-i)^2\,(w^1)^2\overline{w^1} - \frac{\alpha^2+1}{16}\left(1 + A\frac{(\beta-i)(\alpha-i)}{\alpha+\beta}\right)w^1(\overline{w^1})^2 \\
- \frac{\alpha+i}{48}\left(\alpha+i+2A\frac{(\beta-i)(\alpha^2+1)}{\alpha+\beta}\right)(\overline{w^1})^3 + \frac{(\alpha^2+1)(\beta-i)}{2(\alpha+\beta)}\overline{w^1}w^2
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a biholomorphic bijection, hence a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned}
w^1_{ax} &= w^1_a + w^1_x \\
w^2_{ax} &= w^2_a + w^2_x - \frac{\alpha+\beta}{(1+\alpha^2)(1+\beta^2)}(1-\alpha\beta - i(\alpha+\beta)\,\beta^1 x^1 \\
w^3_{ax} &= w^3_a + w^3_x + \frac{1}{2}(\beta^1)^2 x^1 - \beta^2 y^1 + \beta^1 x^1 y^1 - \frac{(\alpha^2+1)(\beta-i)}{\alpha+\beta}\alpha^2 x^1.
\end{aligned}$$

We then get

$$\begin{aligned}
\varphi^1_{ax} &= \varphi^1_a + \varphi^1_x \\
\varphi^2_{ax} &= \varphi^2_a + \varphi^2_x + \frac{\alpha+\beta}{4(\alpha^2+1)(\beta-i)}\left(2(1-i\alpha)\overline{\varphi^1_a} - (\alpha^2+1)\varphi^1_a\right)\varphi^1_x \\
\varphi^3_{ax} &= \varphi^3_a + \varphi^3_x + \chi(a,x)
\end{aligned}$$

where

$$\begin{aligned}
\chi(a,x) = \frac{1}{16(\alpha+\beta)^2}\Big(-(\alpha+\beta)(\alpha+i)^2(\beta+i)\,(\overline{\varphi^1_a})^2 + (\alpha+\beta)(\alpha+2\beta+i)(\alpha-i)^2\,(\varphi^1_a)^2 \\
+4(\alpha+\beta)(\alpha+\beta+i(\alpha\beta-1))\,|\varphi^1_a|^2 - 8(1+\alpha^2)(1+\beta^2)\,\overline{\varphi^2_a} - 8(1+\alpha^2)(\alpha-i)(\beta-i)\,\varphi^2_a\Big)\,\varphi^1_x \\
+ \frac{1}{16}\left(2(\alpha^2-1-2i\alpha)\,\varphi^1_a + (\alpha^2+3+2i)\,\overline{\varphi^1_a}\right)(\varphi^1_x)^2 + \frac{(\alpha^2+1)(\beta-i)}{2(\alpha+\beta)}\overline{\varphi^1_a}\,\varphi^2_x.
\end{aligned}$$

# 6 Lie Algebra $\mathcal{G}_{6,1}$ (isomorphic to $M4$).

Commutation relations for $\mathcal{G}_{6,1}$ : $[x_1, x_2] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = x_6$.

## 6.1 Case $\xi_1^2 \neq 0, \xi_3^2 \neq 0, \xi_3^1 \neq \xi_4^2$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & * & \boxed{\xi_3^1} & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -({\xi_5^5}^2+1)/\xi_5^6 \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{23}$$

where $\mathbf{J_2^1} = (-((\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2-\xi_3^1)\xi_5^5\xi_3^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_1^2))/(\xi_5^6{\xi_3^2}^2)$; $\mathbf{J_4^1} = (-(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1) - \xi_5^6\xi_4^2\xi_3^1))/(\xi_5^6\xi_3^2)$; $\mathbf{J_1^3} = (-(\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - \xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + \xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_4^2}^2{\xi_1^2}^2 + \xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + \xi_4^2{\xi_1^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_3^2\xi_1^2\xi_3^1\xi_1^1))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2)$; $\mathbf{J_2^3} = (-({\xi_5^6}^2\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_2^2 + {\xi_5^6}^2\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_1^1 - {\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1 - {\xi_5^6}^2\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1 - {\xi_5^6}^2\xi_5^5\xi_4^2{\xi_3^2}^2\xi_3^1\xi_1^1 + {\xi_5^6}^2\xi_5^5\xi_4^2\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_5^6}^2{\xi_4^2}^2{\xi_3^2}^2\xi_2^2\xi_1^1 - {\xi_5^6}^2{\xi_4^2}^2{\xi_3^2}^2{\xi_1^1}^2 + {\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2{\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^6}^2{\xi_4^2}^2{\xi_1^2}^2{\xi_3^1}^2 + {\xi_5^6}^2\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - {\xi_5^6}^2\xi_4^2{\xi_3^2}^2\xi_3^1 - {\xi_5^6}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_5^6{\xi_5^5}^3{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6{\xi_5^5}^3\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6{\xi_5^5}^3\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - 2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - \xi_5^6{\xi_5^5}^2{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - \xi_5^6{\xi_5^5}^2{\xi_3^2}^2\xi_3^1 - \xi_5^6{\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - 2\xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 + 2\xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - 2\xi_5^6\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - \xi_5^6{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - \xi_5^6{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^4{\xi_4^2}^2{\xi_1^2}^2 + 2{\xi_5^5}^4\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^4{\xi_1^2}^2{\xi_3^1}^2 - 2{\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + 4{\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - 2{\xi_5^5}^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_4^2}^2{\xi_1^2}^2 + 2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_1^2}^2{\xi_3^1}^2))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_5^6{\xi_3^2}^3)$; $\mathbf{J_3^3} = (-(\xi_5^6\xi_5^5\xi_4^2\xi_3^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^1 + \xi_5^6\xi_4^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_4^2\xi_1^2 + {\xi_5^5}^2\xi_3^2\xi_2^2 + {\xi_5^5}^2\xi_1^2\xi_3^1 - \xi_4^2\xi_1^2 + \xi_3^2\xi_2^2 + \xi_1^2\xi_3^1))/(({\xi_5^5}^2+1)\xi_3^2)$; $\mathbf{J_4^3} = (-({\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2 - {\xi_5^6}^2{\xi_4^2}^2\xi_3^2\xi_1^1 + {\xi_5^6}^2{\xi_4^2}^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_1^2 + \xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2 - \xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^1 + 2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_1^2\xi_3^1 - \xi_5^6{\xi_4^2}^2\xi_1^2 + \xi_5^6\xi_4^2\xi_3^2\xi_2^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^1 + 2\xi_5^6\xi_4^2\xi_1^2\xi_3^1 - {\xi_5^5}^4\xi_4^2\xi_1^2 + {\xi_5^5}^4\xi_1^2\xi_3^1 - 2{\xi_5^5}^2\xi_4^2\xi_1^2 + 2{\xi_5^5}^2\xi_1^2\xi_3^1 - \xi_4^2\xi_1^2 + \xi_1^2\xi_3^1))/(({\xi_5^5}^2+1)\xi_5^6{\xi_3^2}^2)$; $\mathbf{J_1^4} = (-((({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^2-\xi_3^1)\xi_1^2 - ((\xi_3^2{\xi_1^1}^2 + \xi_3^2 - \xi_1^2\xi_3^1\xi_1^1)\xi_3^2 - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2\xi_1^2)\xi_5^6))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_3^2)$; $\mathbf{J_2^4} = (-((({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^2-\xi_3^1)(\xi_3^2\xi_2^2 + \xi_3^2\xi_1^1 - \xi_1^2\xi_3^1) - ((\xi_3^2\xi_2^2\xi_1^1 - \xi_3^2 - \xi_2^2\xi_1^2\xi_3^1)\xi_3^2\xi_3^1 - (\xi_3^2\xi_2^2 + \xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)(\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2)\xi_5^6))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2)$; $\mathbf{J_3^4} = -((\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6 + ({\xi_5^5}^2+1)\xi_1^2)/({\xi_5^5}^2+1)$; $\mathbf{J_4^4} = (-((\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6\xi_4^2 + (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 + \xi_4^2\xi_1^2)({\xi_5^5}^2+1)))/(({\xi_5^5}^2+1)\xi_3^2)$; $\mathbf{J_1^5} = (((\xi_5^5 - \xi_1^1)\xi_1^6 - \xi_2^6\xi_1^2)({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_3^2}^2 + (\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 - 2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - \xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + \xi_5^6{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2 - \xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 + {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + 2{\xi_5^5}^2\xi_4^2{\xi_1^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_5^5}^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_4^2}^2{\xi_1^2}^2 + \xi_4^2\xi_3^2\xi_2^2\xi_1^2 + \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + 2\xi_4^2{\xi_1^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_3^2\xi_1^2\xi_3^1\xi_1^1 - {\xi_1^2}^2{\xi_3^1}^2)\xi_3^6 - ((\xi_3^2{\xi_1^1}^2 + \xi_3^2 - \xi_1^2\xi_3^1\xi_1^1)\xi_3^2 - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1)\xi_4^2\xi_1^2)\xi_5^6\xi_4^6\xi_3^2 + (({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_4^6\xi_3^2 - \xi_3^6\xi_3^1)(\xi_4^2-\xi_3^1)\xi_1^2)/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_5^6{\xi_3^2}^2)$; $\mathbf{J_2^5} = (((({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_1^6\xi_1^2 + \xi_5^6\xi_2^6\xi_5^5{\xi_3^2}^2 - \xi_5^6\xi_2^6{\xi_3^2}^2\xi_2^2)({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1) - (\xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_1^1 - \xi_5^6\xi_5^5{\xi_3^2}^2\xi_2^2\xi_3^1 - \xi_5^6\xi_5^5{\xi_3^2}^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2\xi_2^2\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 + \xi_5^6{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - \xi_5^6{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + {\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_4^2\xi_3^2\xi_2^2\xi_1^2 - \xi_4^2\xi_3^2\xi_1^2\xi_1^1 + \xi_3^2\xi_2^2\xi_1^2\xi_3^1 + \xi_3^2\xi_1^2\xi_3^1\xi_1^1)\xi_5^6\xi_4^6)\xi_3^2 + (\xi_5^6{\xi_5^5}^2{\xi_4^2}^2{\xi_3^2}^2 - 2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2\xi_3^1 + \xi_5^6{\xi_5^5}^2{\xi_3^2}^2{\xi_3^1}^2 + \xi_5^6\xi_5^5{\xi_4^2}^2{\xi_3^2}^2\xi_2^2 - \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1 + \xi_5^6\xi_5^5\xi_4^2{\xi_3^2}^2\xi_3^1\xi_1^1 - \xi_5^6\xi_5^5{\xi_3^2}^2{\xi_3^1}^2\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_3^2}^2\xi_2^2\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_3^2}^2{\xi_1^1}^2 + \xi_5^6{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 + 2\xi_5^6{\xi_4^2}^2\xi_3^2\xi_1^2\xi_3^1\xi_1^1 - \xi_5^6{\xi_4^2}^2{\xi_1^2}^2{\xi_3^1}^2 + \xi_5^6\xi_4^2{\xi_3^2}^2\xi_2^2\xi_3^1\xi_1^1 - \xi_5^6\xi_4^2{\xi_3^2}^2\xi_3^1 - \xi_5^6\xi_4^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^3{\xi_4^2}^2\xi_3^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - {\xi_5^5}^3\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_5^5}^2{\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + {\xi_5^5}^2\xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + {\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + 2{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - {\xi_5^5}^2\xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_5^5}^2{\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - {\xi_5^5}^2{\xi_3^2}^2\xi_3^1 - {\xi_5^5}^2\xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + {\xi_5^5}^2\xi_3^2\xi_1^2{\xi_3^1}^2\xi_1^1 - \xi_5^5{\xi_4^2}^2\xi_3^2\xi_1^2 + 2\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^5\xi_3^2\xi_1^2{\xi_3^1}^2 - {\xi_4^2}^2\xi_3^2\xi_2^2\xi_1^2 - {\xi_4^2}^2\xi_3^2\xi_1^2\xi_1^1 + {\xi_4^2}^2{\xi_1^2}^2\xi_3^1 + \xi_4^2{\xi_3^2}^2{\xi_2^2}^2 + \xi_4^2{\xi_3^2}^2 + 2\xi_4^2\xi_3^2\xi_2^2\xi_1^2\xi_3^1 - \xi_4^2{\xi_1^2}^2{\xi_3^1}^2 - {\xi_3^2}^2{\xi_2^2}^2\xi_3^1 - {\xi_3^2}^2\xi_3^1 - \xi_3^2\xi_2^2\xi_1^2{\xi_3^1}^2 + \xi_3^2\xi_1^2{\xi_3^1}^2\xi_1^1)\xi_5^6\xi_3^6 + (\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2-\xi_3^1)\xi_5^5\xi_3^2)({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_5^6\xi_1^6\xi_3^2 - (({\xi_5^5}^2+1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^2)(\xi_5^6\xi_4^6\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6\xi_3^6\xi_5^5\xi_4^2\xi_3^2 + \xi_5^6\xi_3^6\xi_5^5\xi_3^2\xi_3^1 + \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_1^1 - \xi_5^6\xi_3^6\xi_4^2\xi_1^2\xi_3^1 - \xi_5^6\xi_3^6\xi_3^2\xi_3^1\xi_1^1 + \xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2 - \xi_3^6{\xi_5^5}^2\xi_1^2\xi_3^1 + \xi_3^6\xi_4^2\xi_1^2 - \xi_3^6\xi_1^2\xi_3^1)(\xi_4^2-\xi_3^1))/(({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1){\xi_5^6}^2{\xi_3^2}^3)$; $\mathbf{J_3^5} = (-(((\xi_2^6\xi_3^2 + \xi_1^6\xi_3^1 - \xi_3^6\xi_5^5 - \xi_4^6\xi_1^2)({\xi_5^5}^2+1) - (\xi_3^2\xi_1^1 - \xi_1^2\xi_3^1 - \xi_5^5\xi_3^2)\xi_5^6\xi_4^6)\xi_3^2 + (\xi_4^2\xi_3^2\xi_1^1 - \xi_4^2\xi_1^2\xi_3^1 - \xi_3^2\xi_2^2\xi_3^1 - (\xi_4^2-\xi_3^1)\xi_5^5\xi_3^2)\xi_5^6\xi_3^6 - ((({\xi_5^5}^2+1)\xi_2^2 + (\xi_5^5-\xi_2^2)\xi_5^6\xi_3^1)\xi_3^2 - ({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)\xi_1^2)\xi_3^6))/(({\xi_5^5}^2+1)\xi_5^6\xi_3^2)$; $\mathbf{J_4^5} = (-({\xi_5^6}^2\xi_4^6\xi_5^5\xi_4^2{\xi_3^2}^2 - {\xi_5^6}^2\xi_4^6\xi_4^2{\xi_3^2}^2\xi_1^1 + {\xi_5^6}^2\xi_4^6\xi_4^2\xi_3^2\xi_1^2\xi_3^1 - {\xi_5^6}^2\xi_3^6\xi_5^5{\xi_4^2}^2\xi_3^2 + {\xi_5^6}^2\xi_3^6{\xi_4^2}^2\xi_3^2\xi_1^1 - {\xi_5^6}^2\xi_3^6{\xi_4^2}^2\xi_1^2\xi_3^1 - \xi_5^6\xi_4^6{\xi_5^5}^3{\xi_3^2}^2 - \xi_5^6\xi_4^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2 - \xi_5^6\xi_4^6{\xi_5^5}^2{\xi_3^2}^2\xi_1^1 + \xi_5^6\xi_4^6{\xi_5^5}^2\xi_3^2\xi_1^2\xi_3^1 - \xi_5^6\xi_4^6\xi_5^5{\xi_3^2}^2 - \xi_5^6\xi_4^6\xi_4^2\xi_3^2\xi_1^2 - \xi_5^6\xi_4^6{\xi_3^2}^2\xi_1^1 + \xi_5^6\xi_4^6\xi_3^2\xi_1^2\xi_3^1 + \xi_5^6\xi_3^6{\xi_5^5}^2{\xi_4^2}^2\xi_1^2 - \xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^1 - 2\xi_5^6\xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2\xi_3^1 + \xi_5^6\xi_3^6{\xi_4^2}^2\xi_1^2 - \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_2^2 + \xi_5^6\xi_3^6\xi_4^2\xi_3^2\xi_1^1 - 2\xi_5^6\xi_3^6\xi_4^2\xi_1^2\xi_3^1 + \xi_5^6\xi_2^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2 + \xi_5^6\xi_2^6\xi_4^2{\xi_3^2}^2 + \xi_5^6\xi_1^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_3^1 + \xi_5^6\xi_1^6\xi_4^2\xi_3^2\xi_3^1 + \xi_3^6{\xi_5^5}^4\xi_4^2\xi_1^2 - \xi_3^6{\xi_5^5}^4\xi_1^2\xi_3^1 + 2\xi_3^6{\xi_5^5}^2\xi_4^2\xi_1^2 - 2\xi_3^6{\xi_5^5}^2\xi_1^2\xi_3^1 + \xi_3^6\xi_4^2\xi_1^2 - \xi_3^6\xi_1^2\xi_3^1 - \xi_1^6{\xi_5^5}^4\xi_4^2\xi_3^2 + \xi_1^6{\xi_5^5}^4\xi_3^2\xi_3^1 - 2\xi_1^6{\xi_5^5}^2\xi_4^2\xi_3^2 + 2\xi_1^6{\xi_5^5}^2\xi_3^2\xi_3^1 - \xi_1^6\xi_4^2\xi_3^2 + \xi_1^6\xi_3^2\xi_3^1))/(({\xi_5^5}^2+1){\xi_5^6}^2{\xi_3^2}^2)$;
and the parameters are subject to the condition

$$\xi_1^2\xi_3^2\xi_5^6(\xi_3^1 - \xi_4^2) \neq 0. \tag{24}$$

Now the automorphism group of $\mathcal{G}_{6,1}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_1^1 u & -b_2^1 u & 0 & 0 \\ b_1^4 & b_2^4 & -b_1^2 u & b_2^2 u & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & b_2^2 b_1^1 - b_1^2 b_2^1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & b_2^3 b_1^2 - b_1^3 b_2^2 - b_1^4 b_2^1 + b_2^4 b_1^1 & (b_2^2 b_1^1 - b_1^2 b_2^1) u \end{pmatrix} \tag{25}$$

where $u \in \mathbb{R}$, $u \neq 0$ and $b_2^2 b_1^1 - b_1^2 b_2^1 \neq 0$. Taking suitable values for $u$ and the $b_j^i$'s, equivalence by $\Phi$ leads to the case $\xi_1^1 = \xi_3^1 = \xi_2^2 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$ and $\xi_1^2 = \xi_3^2 = \xi_5^6 = 1, \xi_4^2 = \alpha \neq 0$:

$$J_\alpha = \begin{pmatrix} 0 & -\alpha & 0 & -\alpha & 0 & 0 \\ 1 & 0 & 1 & \alpha & 0 & 0 \\ \alpha - 1 & \alpha - 1 & \alpha & (\alpha+1)\alpha & 0 & 0 \\ -(\alpha-1)/\alpha & 0 & -1 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad (\alpha \neq 0). \tag{26}$$

The $J_\alpha$'s corresponding to distinct $\alpha$'s are not equivalent.

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_2] = (1-\alpha)\tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_3] = -\tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_4] = -\alpha(\tilde{x}_5 + \tilde{x}_6)$; $[\tilde{x}_2, \tilde{x}_3] = \alpha\tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_4] = \alpha(\alpha\tilde{x}_5 - \tilde{x}_6)$; $[\tilde{x}_3, \tilde{x}_4] = -\alpha\tilde{x}_5$.

## 6.2 Case $\xi_1^2 \neq 0, \xi_3^2 = 0, \xi_3^1 \neq \xi_4^2$.

$$J = \begin{pmatrix} -\frac{\xi_1^4 \xi_4^2 + \xi_2^2 \xi_1^2}{\xi_1^2} & -\frac{{\xi_2^2}^2 + 1 + \xi_2^4 \xi_4^2}{\xi_1^2} & \boxed{\xi_3^1} & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & 0 & \boxed{\xi_4^2} & 0 & 0 \\ * & * & -\frac{\xi_5^5 \xi_4^2 - \xi_5^5 \xi_3^1 + \xi_2^2 \xi_3^1}{\xi_4^2} & * & 0 & 0 \\ \boxed{\xi_1^4} & \boxed{\xi_2^4} & -\frac{\xi_1^2 \xi_3^1}{\xi_4^2} & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & \frac{\xi_4^2 \xi_3^1}{\xi_4^2 - \xi_3^1} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)}{\xi_4^2 \xi_3^1} & -\xi_5^5 \end{pmatrix} \tag{27}$$

where $\mathbf{J_4^1} = (-((\xi_4^2 + \xi_3^1)\xi_2^2\xi_1^2 + \xi_1^4{\xi_4^2}^2 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^2))/{\xi_1^2}^2$; $\mathbf{J_1^3} = (-((\xi_4^2 - \xi_3^1)\xi_1^4\xi_2^2 - \xi_2^4\xi_4^2\xi_1^2 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^4))/(\xi_1^2\xi_3^1)$; $\mathbf{J_2^3} = ((\xi_4^2 + \xi_3^1)\xi_2^4\xi_2^2\xi_1^2 - ({\xi_2^2}^2 + 1)\xi_1^4\xi_4^2 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^4\xi_1^2)/({\xi_1^2}^2\xi_3^1)$; $\mathbf{J_4^3} = ((((\xi_4^2 - \xi_3^1){\xi_5^5}^2 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^2 + ({\xi_2^2}^2 + 1)\xi_4^2 + \xi_2^4{\xi_4^2}^2)(\xi_4^2 - \xi_3^1) + ({\xi_2^2}^2 + 1 + \xi_2^4\xi_4^2)\xi_4^2\xi_3^1)\xi_1^2 + ((\xi_4^2 + \xi_3^1)\xi_2^2\xi_1^2 + \xi_1^4{\xi_4^2}^2)(\xi_5^5 - \xi_2^2)(\xi_4^2 - \xi_3^1))/(\xi_4^2{\xi_1^2}^2\xi_3^1)$; $\mathbf{J_4^4} = (\xi_1^4{\xi_4^2}^2 + \xi_2^2\xi_1^2\xi_3^1 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^2)/(\xi_4^2\xi_1^2)$; $\mathbf{J_1^5} = (-(\xi_4^6\xi_1^4\xi_1^2\xi_3^1 + \xi_3^6\xi_5^5\xi_1^4\xi_4^2 - \xi_3^6\xi_5^5\xi_1^4\xi_3^1 + \xi_3^6\xi_2^4\xi_4^2\xi_1^2 - \xi_3^6\xi_1^4\xi_4^2\xi_2^2 + \xi_3^6\xi_1^4\xi_2^2\xi_3^1 + \xi_2^6{\xi_1^2}^2\xi_3^1 - \xi_1^6\xi_5^5\xi_1^2\xi_3^1 - \xi_1^6\xi_1^4\xi_4^2\xi_3^1 - \xi_1^6\xi_2^2\xi_1^2\xi_3^1)\xi_4^2)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)\xi_1^2)$; $\mathbf{J_2^5} = (-((\xi_4^6\xi_2^4\xi_1^2 - \xi_2^6\xi_5^5\xi_1^2 + \xi_2^6\xi_2^2\xi_1^2 - \xi_1^6\xi_2^4\xi_4^2 - \xi_1^6{\xi_2^2}^2 - \xi_1^6)\xi_1^2\xi_3^1 + (\xi_5^5\xi_2^4\xi_1^2 + \xi_2^4\xi_2^2\xi_1^2 - \xi_1^4{\xi_2^2}^2 - \xi_1^4)(\xi_4^2 - \xi_3^1)\xi_3^6 - (({\xi_2^2}^2 + 1)\xi_1^4 - 2\xi_2^4\xi_2^2\xi_1^2)\xi_3^6\xi_3^1)\xi_4^2)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1){\xi_1^2}^2)$; $\mathbf{J_3^5} = ((\xi_4^6\xi_1^2\xi_3^1 + 2\xi_3^6\xi_5^5\xi_4^2 - \xi_3^6\xi_5^5\xi_3^1 + \xi_3^6\xi_2^2\xi_3^1 - \xi_1^6\xi_4^2\xi_3^1)\xi_3^1)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1))$; $\mathbf{J_4^5} = (((\xi_4^2 - \xi_3^1)\xi_4^6\xi_1^2 + \xi_1^6\xi_4^2\xi_3^1 - (\xi_5^5 - \xi_2^2)(\xi_4^2 - \xi_3^1)\xi_3^6)((\xi_4^2 + \xi_3^1)\xi_2^2\xi_1^2 + \xi_1^4{\xi_4^2}^2) + ((\xi_4^6\xi_5^5 - \xi_2^6\xi_4^2)\xi_1^2 + (\xi_4^2 - \xi_3^1)\xi_1^6\xi_5^5)\xi_4^2\xi_1^2\xi_3^1 - ((\xi_1^4\xi_4^2 + \xi_2^2\xi_1^2)(\xi_4^2 - \xi_3^1)\xi_4^2 + (\xi_1^4\xi_4^2 + \xi_2^2\xi_1^2)\xi_4^2\xi_3^1 + (\xi_4^2 - \xi_3^1)\xi_5^5\xi_1^2\xi_3^1)\xi_4^6\xi_1^2 - (((\xi_4^2 - \xi_3^1){\xi_5^5}^2 - (\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^2 + ({\xi_2^2}^2 + 1)\xi_4^2 + \xi_2^4{\xi_4^2}^2)(\xi_4^2 - \xi_3^1) + ({\xi_2^2}^2 + 1 + \xi_2^4\xi_4^2)\xi_4^2\xi_3^1)\xi_3^6\xi_1^2)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1){\xi_1^2}^2)$; and the parameters are subject to the condition

$$\xi_1^2 \xi_4^2 \xi_3^1 (\xi_3^1 - \xi_4^2) \neq 0. \tag{28}$$

Taking $u = 1$ and suitable values for the $b_j^i$'s in (25), equivalence by $\Phi$ switches to the case 6.5 $\xi_1^2 = \xi_3^2 = 0$ below.

## 6.3 Case $\xi_1^2 \neq 0, \xi_3^1 = \xi_4^2$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & \boxed{\xi_2^1} & 0 & 0 & 0 & 0 \\ -\frac{{\xi_1^1}^2 + 1}{\xi_2^1} & -\xi_1^1 & 0 & 0 & 0 & 0 \\ \xi_2^4 & \frac{(2\xi_2^4\xi_1^1 - \xi_1^4\xi_2^1)\xi_2^1}{{\xi_1^1}^2 + 1} & \xi_1^1 & -\xi_2^1 & 0 & 0 \\ \boxed{\xi_1^4} & \boxed{\xi_2^4} & \frac{{\xi_1^1}^2 + 1}{\xi_2^1} & -\xi_1^1 & 0 & 0 \\ * & * & \frac{(\xi_5^5 - \xi_1^1)\xi_3^6\xi_2^1 - ({\xi_1^1}^2 + 1)\xi_4^6}{\xi_5^6\xi_2^1} & \frac{\xi_4^6\xi_5^5 + \xi_4^6\xi_1^1 + \xi_3^6\xi_2^1}{\xi_5^6} & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2 + 1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{29}$$

where $\mathbf{J}^{5}_{1} = ((\xi_5^5 - \xi_1^1)\xi_1^6\xi_2^1 + ({\xi_1^1}^2 + 1)\xi_2^6 - \xi_3^6\xi_2^4\xi_2^1 - \xi_4^6\xi_1^4\xi_2^1)/(\xi_5^6\xi_2^1)$; $\mathbf{J}^{5}_{2} = (-((2\xi_2^4\xi_1^1 - \xi_1^4\xi_2^1)\xi_3^6\xi_2^1 + ({\xi_1^1}^2 + 1)\xi_4^6\xi_2^4 - (\xi_2^6\xi_5^5 + \xi_2^6\xi_1^1 - \xi_1^6\xi_2^1)({\xi_1^1}^2 + 1)))/(({\xi_1^1}^2 + 1)\xi_5^6)$; and the parameters are subject to the condition

$$\xi_2^1\xi_5^6 \neq 0. \tag{30}$$

Taking $u = \frac{\xi_5^6}{1+{\xi_5^5}^2}$ and suitable values for the $b_j^i$'s in (25), equivalence by $\Phi$ leads to the case where $\xi_1^1 = \xi_1^4 = \xi_2^4 = \xi_5^5 = \xi_1^6 = \xi_2^6 = \xi_3^6 = \xi_4^6 = 0$ and $\xi_2^1 = \xi_5^6 = 1$:

$$J = \mathrm{diag}\left( \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right) . \tag{31}$$

$J$ is not equivalent to any $J_\alpha$. $\mathfrak{m}$ is here an abelian algebra.

## 6.4 Case $\xi_1^2 = 0, \xi_3^2 \neq 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & * & * & * & 0 & 0 \\ 0 & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ -\frac{\xi_1^4\xi_4^2}{\xi_3^2} & * & -\frac{\xi_5^6\xi_5^5\xi_4^2 - \xi_5^6\xi_4^2\xi_1^1 + {\xi_5^5}^2\xi_2^2 + \xi_2^2}{{\xi_5^5}^2+1} & * & 0 & 0 \\ \boxed{\xi_1^4} & * & \frac{(\xi_5^5-\xi_1^1)\xi_5^6\xi_3^2}{{\xi_5^5}^2+1} & -\frac{({\xi_5^5}^2+1)\xi_1^1 - (\xi_5^5-\xi_1^1)\xi_5^6\xi_4^2}{{\xi_5^5}^2+1} & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{32}$$

where $\mathbf{J}^{1}_{2} = (({\xi_5^5}^2+1)(\xi_2^2-\xi_1^1)\xi_1^4\xi_4^2 + (\xi_5^5-\xi_2^2)({\xi_1^1}^2+1)\xi_5^6\xi_3^2)/(({\xi_5^5}^2+1)\xi_1^4\xi_3^2)$; $\mathbf{J}^{1}_{3} = (({\xi_5^5}^2+1)\xi_1^4\xi_4^2 - ({\xi_1^1}^2+1)\xi_5^6\xi_3^2)/(({\xi_5^5}^2+1)\xi_1^4)$; $\mathbf{J}^{1}_{4} = (-(({\xi_5^5}^2 + 1)({\xi_1^1}^2 + 1)\xi_3^2 - ({\xi_5^5}^2 + 1)\xi_1^4{\xi_4^2}^2 + ({\xi_1^1}^2 + 1)\xi_5^6\xi_4^2\xi_3^2))/(({\xi_5^5}^2 + 1)\xi_1^4\xi_3^2)$; $\mathbf{J}^{3}_{2} = (-((({\xi_2^2}^2 + 1)\xi_3^2 - \xi_1^4{\xi_4^2}^2)({\xi_5^5}^2 + 1) - (\xi_2^2\xi_1^1 - 1 - (\xi_2^2 + \xi_1^1)\xi_5^5)\xi_5^6\xi_4^2\xi_3^2))/(({\xi_5^5}^2 + 1){\xi_3^2}^2)$; $\mathbf{J}^{3}_{4} = (-(\xi_5^6\xi_5^5\xi_4^2 - \xi_5^6\xi_4^2\xi_1^1 + {\xi_5^5}^2\xi_2^2 - {\xi_5^5}^2\xi_1^1 + \xi_2^2 - \xi_1^1)\xi_4^2)/(({\xi_5^5}^2 + 1)\xi_3^2)$; $\mathbf{J}^{4}_{2} = (-((\xi_2^2\xi_1^1 - 1 - (\xi_2^2 + \xi_1^1)\xi_5^5)\xi_5^6\xi_3^2 + ({\xi_5^5}^2 + 1)\xi_1^4\xi_4^2))/(({\xi_5^5}^2 + 1)\xi_3^2)$; $\mathbf{J}^{5}_{1} = (-(\xi_4^6\xi_1^4\xi_3^2 - \xi_3^6\xi_1^4\xi_4^2 - \xi_1^6\xi_5^5\xi_3^2 + \xi_1^6\xi_3^2\xi_1^1))/(\xi_5^6\xi_3^2)$; $\mathbf{J}^{5}_{2} = ((({\xi_5^5}^2+1)(\xi_5^5-\xi_2^2)\xi_2^6\xi_1^4\xi_3^2 - ({\xi_5^5}^2+1)(\xi_2^2-\xi_1^1)\xi_1^6\xi_1^4\xi_4^2 - (\xi_5^5-\xi_2^2)({\xi_1^1}^2+1)\xi_5^6\xi_1^6\xi_3^2 - ((\xi_2^2+\xi_1^1)\xi_5^5 + {\xi_1^1}^2+1)\xi_5^6\xi_4^6\xi_1^4\xi_3^2 - (\xi_2^2\xi_1^1 - 1 - (\xi_2^2+\xi_1^1)\xi_5^5)\xi_5^6\xi_3^6\xi_1^4\xi_4^2)\xi_3^2 + (({\xi_2^2}^2+1)\xi_3^2 - \xi_1^4{\xi_4^2}^2)({\xi_5^5}^2+1)\xi_3^6\xi_1^4 + (({\xi_5^5}^2+1)\xi_1^4\xi_4^2 + (\xi_2^2+\xi_1^1)\xi_5^6\xi_3^2\xi_1^1)\xi_4^6\xi_1^4\xi_3^2)/(({\xi_5^5}^2+1)\xi_5^6\xi_1^4{\xi_3^2}^2)$; $\mathbf{J}^{5}_{3} = (-((((\xi_5^5-\xi_1^1)\xi_4^6\xi_1^4 - ({\xi_1^1}^2+1)\xi_1^6)\xi_3^2 - (\xi_2^2-\xi_1^1)\xi_3^6\xi_1^4\xi_4^2)\xi_5^6 + (\xi_2^6\xi_3^2 + \xi_1^6\xi_4^2 - \xi_3^6\xi_5^5)({\xi_5^5}^2+1)\xi_1^4 - (({\xi_5^5}^2+1)\xi_2^2 + (\xi_5^5-\xi_2^2)\xi_5^6\xi_4^2)\xi_3^6\xi_1^4))/(({\xi_5^5}^2+1)\xi_5^6\xi_1^4)$; $\mathbf{J}^{5}_{4} = (-(((((\xi_5^5 - \xi_1^1)\xi_4^6\xi_1^4 - ({\xi_1^1}^2 + 1)\xi_1^6)\xi_5^6\xi_4^2 - ({\xi_5^5}^2 + 1)({\xi_1^1}^2 + 1)\xi_1^6)\xi_3^2 + ((\xi_2^6\xi_3^2 + \xi_1^6\xi_4^2)\xi_4^2 - (\xi_5^5 + \xi_1^1)\xi_4^6\xi_3^2)({\xi_5^5}^2 + 1)\xi_1^4 + (({\xi_1^1}^2 + 1)\xi_3^2 - \xi_1^4{\xi_4^2}^2)(\xi_5^5 - \xi_2^2)\xi_5^6\xi_3^6 - (({\xi_5^5}^2 + 1)(\xi_2^2 - \xi_1^1)\xi_1^4\xi_4^2 + (\xi_5^5 - \xi_2^2)({\xi_1^1}^2 + 1)\xi_5^6\xi_3^2 + (\xi_2^2 - \xi_1^1)\xi_5^6\xi_1^4{\xi_4^2}^2)\xi_3^6))/(({\xi_5^5}^2 + 1)\xi_5^6\xi_1^4\xi_3^2)$;
and the parameters are subject to the condition

$$\xi_3^2\xi_1^4\xi_5^6 \neq 0. \tag{33}$$

Taking suitable values for $u$ and the $b_j^i$'s in (25), equivalence by $\Phi$ switches to the case 6.1, more precisely $\xi_1^2 = \xi_3^2 = \xi_4^2 = 1, \xi_3^1 = 0$. Hence $J$ is equivalent to $J_\alpha$ in (26) with $\alpha = 1$.

## 6.5 Case $\xi_1^2 = 0, \xi_3^2 = 0$.

$$J = \begin{pmatrix} \frac{\xi_5^5\xi_4^2 - \xi_5^5\xi_3^1 + \xi_2^2\xi_3^1}{\xi_4^2} & \boxed{\xi_2^1} & \boxed{\xi_3^1} & \frac{(\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1 + \xi_2^3\xi_4^2\xi_3^1 + (\xi_4^2-\xi_3^1)\xi_5^5\xi_2^1}{{\xi_2^2}^2+1} & 0 & 0 \\ 0 & \boxed{\xi_2^2} & 0 & \boxed{\xi_4^2} & 0 & 0 \\ * & \boxed{\xi_2^3} & -\frac{\xi_5^5\xi_4^2 - \xi_5^5\xi_3^1 + \xi_2^2\xi_3^1}{\xi_4^2} & * & 0 & 0 \\ 0 & -\frac{{\xi_2^2}^2+1}{\xi_4^2} & 0 & -\xi_2^2 & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{\xi_4^2\xi_3^1}{\xi_4^2-\xi_3^1} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{({\xi_5^5}^2+1)(\xi_4^2-\xi_3^1)}{\xi_4^2\xi_3^1} & -\xi_5^5 \end{pmatrix} \tag{34}$$

where $\mathbf{J}^{3}_{1} = (-((\xi_4^2 - \xi_3^1){\xi_5^5}^2\xi_4^2 - (\xi_4^2 - \xi_3^1){\xi_5^5}^2\xi_3^1 + 2(\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^2\xi_3^1 + ({\xi_2^2}^2 + 1)\xi_4^2\xi_3^1 + (\xi_4^2 - {\xi_2^2}^2\xi_3^1)(\xi_4^2 - \xi_3^1)))/({\xi_4^2}^2\xi_3^1)$; $\mathbf{J}^{3}_{4} = (-((((\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1 + \xi_2^3\xi_4^2\xi_3^1)(\xi_4^2-\xi_3^1)\xi_5^5 - ((\xi_4^2+\xi_3^1)\xi_2^2\xi_2^1 + \xi_2^3\xi_4^2\xi_3^1)(\xi_4^2-\xi_3^1)\xi_2^2 + ((\xi_4^2-\xi_3^1)^2{\xi_5^5}^2 - (\xi_4^2-\xi_3^1)^2\xi_5^5\xi_2^2 + (\xi_4^2-\xi_3^1)({\xi_2^2}^2+1)\xi_4^2 + ({\xi_2^2}^2+1)\xi_4^2\xi_3^1)\xi_2^1))/(({\xi_2^2}^2+1)\xi_4^2\xi_3^1)$; $\mathbf{J}^{5}_{1} = ((\xi_5^5 - \xi_2^2)\xi_1^6\xi_4^2{\xi_3^1}^2 + (\xi_4^2 - {\xi_2^2}^2\xi_3^1)(\xi_4^2 - \xi_3^1)\xi_3^6 + ((\xi_4^2 - \xi_3^1){\xi_5^5}^2\xi_4^2 - (\xi_4^2 - \xi_3^1){\xi_5^5}^2\xi_3^1 + 2(\xi_4^2 - \xi_3^1)\xi_5^5\xi_2^2\xi_3^1 + ({\xi_2^2}^2 + 1)\xi_4^2\xi_3^1)\xi_3^6)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1)\xi_4^2)$; $\mathbf{J}^{5}_{2} = (((\xi_2^6\xi_5^5 - \xi_2^6\xi_2^2 - \xi_1^6\xi_2^1 - \xi_3^6\xi_2^3)\xi_4^2 + ({\xi_2^2}^2 + 1)\xi_4^6)\xi_3^1)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1))$; $\mathbf{J}^{5}_{3} = ((2\xi_3^6\xi_5^5\xi_4^2 - \xi_3^6\xi_5^5\xi_3^1 + \xi_3^6\xi_2^2\xi_3^1 - \xi_1^6\xi_4^2\xi_3^1)\xi_3^1)/(({\xi_5^5}^2 + 1)(\xi_4^2 - \xi_3^1))$; $\mathbf{J}^{5}_{4} = (((\xi_4^2 - \xi_3^1)^2{\xi_5^5}^2 - (\xi_4^2 - \xi_3^1)^2\xi_5^5\xi_2^2 + (\xi_4^2 - \xi_3^1)({\xi_2^2}^2 +$

$1)\xi^2_4 + ({\xi^2_2}^2 + 1)\xi^2_4\xi^1_3)\xi^6_3\xi^1_2 + (\xi^6_4\xi^5_5{\xi^2_2}^2 + \xi^6_4\xi^5_5 + \xi^6_4{\xi^2_2}^3 + \xi^6_4\xi^2_2 - \xi^6_2\xi^2_4{\xi^2_2}^2 - \xi^6_2\xi^2_4 - \xi^6_1\xi^5_5\xi^2_4\xi^1_2 + \xi^6_1\xi^5_5\xi^1_3\xi^1_2)\xi^2_4\xi^1_3 + ((\xi^2_4 + \xi^1_3)\xi^2_2\xi^1_2 + \xi^3_2\xi^2_4\xi^1_3)((\xi^2_4 - \xi^1_3)\xi^6_3\xi^5_5 - (\xi^2_4 - \xi^1_3)\xi^6_3\xi^2_2 - \xi^6_1\xi^2_4\xi^1_3))/(({\xi^5_5}^2 + 1)(\xi^2_4 - \xi^1_3)({\xi^2_2}^2 + 1));$ and the parameters are subject to the condition

$$\xi^1_3\xi^2_4(\xi^2_4 - \xi^1_3) \neq 0. \tag{35}$$

• Suppose first $\xi^2_4 \neq -\xi^1_3$, taking suitable values for $u$ and the $b^i_j$'s in (25), equivalence by $\Phi$ leads to the case where $\xi^1_2 = \xi^2_2 = \xi^3_2 = \xi^5_5 = \xi^6_1 = \xi^6_2 = \xi^6_3 = \xi^6_4 = 0$ and $\xi^1_3 = 1, \xi^2_4 = \beta \neq 0, \pm 1$:

$$J'_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta/(\beta-1) \\ 0 & 0 & 0 & 0 & (\beta-1)/\beta & 0 \end{pmatrix} \qquad (\beta \neq 0, \pm 1). \tag{36}$$

The $J$'s corresponding to different $\beta, \beta'$ are not equivalent unless $\beta' = 1/\beta$. $J'_\beta$ is equivalent to $J_\alpha$ in (26) if and only if $\alpha = \frac{(\beta-1)^2}{\beta}$.

• Suppose now $\xi^2_4 = -\xi^1_3$ in (34). Taking suitable values for $u$ and the $b^i_j$'s in (25), equivalence by $\Phi$ leads to the case where $\xi^1_2 = \xi^2_2 = \xi^5_5 = \xi^6_1 = \xi^6_2 = \xi^6_3 = \xi^6_4 = 0$ and $\xi^1_3 = 1, \xi^2_4 = -1, \xi^3_2 = \gamma:$

$$J''_\gamma = \begin{pmatrix} 0 & 0 & 1 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \qquad (\gamma \in \mathbb{R}). \tag{37}$$

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_2] = \tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = 2\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = 2\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = -2\gamma\tilde{x}_6$; $[\tilde{x}_3, \tilde{x}_4] = \tilde{x}_5$.

$J''_\gamma$ is not equivalent to any $J_\alpha$ in (26) nor to (31), and each $J''_\gamma$ ($\gamma \neq 0$) is equivalent to $J''_1$. $J''_1$ is not equivalent to $J''_0$.

## 6.6 Conclusions.

One has with obvious notations

$$\mathfrak{X}_{6,1} = \mathfrak{X}_{\xi^2_1 \neq 0} \cup \mathfrak{X}_{\xi^2_3 \neq 0} \cup \mathfrak{X}_{\xi^2_4 \neq 0} \tag{38}$$

and

$$\mathfrak{X}_{\xi^2_3 \neq 0} \subset \mathfrak{X}_{\xi^1_3 \neq \xi^2_4}.$$

It can be seen that the formula (23), which still makes sense for $\xi^6_5 \neq 0$ under the only assumption that $\xi^2_3 \neq 0$, yields all of $\mathfrak{X}_{\xi^2_3 \neq 0}$. Hence $\mathfrak{X}_{\xi^2_3 \neq 0}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$ with a global chart. The automorphism $\Phi = \operatorname{diag}(1, 1, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), 1, 1)$ switches $\xi^2_3$ and $\xi^2_4$, hence $\mathfrak{X}_{\xi^2_4 \neq 0} = \Phi\, \mathfrak{X}_{\xi^2_3 \neq 0}\, \Phi^{-1}$ is also a 12-dimensional submanifold of $\mathbb{R}^{36}$. Consider now $\mathfrak{X}_{\xi^2_1 \neq 0}$. There are 3 subcases:

$$\xi^2_3 \neq 0 \qquad \text{(which implies } \xi^1_3 \neq \xi^2_4 \text{ from section 6.3)} \tag{39}$$

$$\xi^2_3 = 0 \qquad \text{and} \qquad \xi^1_3 \neq \xi^2_4 \tag{40}$$

$$\xi^2_3 = 0 \qquad \text{and} \qquad \xi^1_3 = \xi^2_4. \tag{41}$$

To prove that $\mathfrak{X}_{\xi^2_1 \neq 0}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$, it is sufficient to prove that it is a local submanifold in the neighborhood of any of its points. Take any $K \in \mathfrak{X}_{\xi^2_1 \neq 0} : K = (\xi^i_j(K))$. In case (39), $K \in \mathfrak{X}_{\xi^2_3 \neq 0}$ and then, from the section 6.1, $\mathfrak{X}_{\xi^2_1 \neq 0}$ is a local 12-dimensional submanifold of $\mathbb{R}^{36}$. Suppose now $K$ belongs in case (40) or (41). To solve the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ in the neighborhood of $K$, one has to complete first a set of common steps, and then we are left with solving the system $S$ of the

remaining equations in the 15 variables $\xi^1_1, \xi^1_2, \xi^1_3, \xi^2_1, \xi^2_2, \xi^2_3, \xi^2_4, \xi^4_1, \xi^4_2, \xi^5_5, \xi^6_1, \xi^6_2, \xi^6_3, \xi^6_4, \xi^6_5$ in the open subset $\xi^6_5 \neq 0$ of $\mathbb{R}^{15}$. Among these equations, we single out the 3 following equations :

$$\begin{cases} f = 0 \\ g = 0 \\ h = 0 \end{cases} \tag{42}$$

where : $f = {J^2}^2_1 = (-\xi^6_5\xi^5_5\xi^2_3\xi^2_2 - \xi^6_5\xi^5_5\xi^2_3\xi^1_1 + \xi^6_5\xi^2_3\xi^2_2\xi^1_1 - \xi^6_5\xi^2_3\xi^2_1\xi^1_2 - \xi^6_5\xi^2_3 + {\xi^5_5}^2\xi^4_2\xi^2_3 + {\xi^5_5}^2\xi^4_1\xi^2_4 + {\xi^5_5}^2\xi^2_2\xi^2_1 + {\xi^5_5}^2\xi^2_1\xi^1_1 + \xi^4_2\xi^2_3 + \xi^4_1\xi^2_4 + \xi^2_2\xi^2_1 + \xi^2_1\xi^1_1)/({\xi^5_5}^2 + 1)$; $g = {J^2}^2_2 = (-\xi^6_5\xi^5_5\xi^2_3{\xi^2_2}^2 - \xi^6_5\xi^5_5\xi^2_3\xi^2_1\xi^1_2 - \xi^6_5\xi^5_5\xi^2_3 + \xi^6_5\xi^2_3{\xi^2_2}^2\xi^1_1 - \xi^6_5\xi^2_3\xi^2_2\xi^2_1\xi^1_2 + \xi^6_5\xi^2_3\xi^1_1 + {\xi^5_5}^2\xi^4_2\xi^2_4\xi^2_1 + {\xi^5_5}^2\xi^4_2\xi^2_3\xi^2_2 - {\xi^5_5}^2\xi^4_2\xi^2_3\xi^1_1 + {\xi^5_5}^2\xi^4_1\xi^2_3\xi^1_2 + {\xi^5_5}^2{\xi^2_2}^2\xi^2_1 + {\xi^5_5}^2{\xi^2_1}^2\xi^1_2 + \xi^4_2\xi^2_4\xi^2_1 + \xi^4_2\xi^2_3\xi^2_2 - \xi^4_2\xi^2_3\xi^1_1 + \xi^4_1\xi^2_3\xi^1_2 + {\xi^2_2}^2\xi^2_1 + {\xi^2_1}^2\xi^1_2)/(\xi^2_1({\xi^5_5}^2 + 1))$; $h = {J^2}^2_3 = (\xi^6_5\xi^5_5\xi^2_4\xi^2_3 - \xi^6_5\xi^5_5\xi^2_3\xi^1_3 - \xi^6_5\xi^2_4\xi^2_3\xi^1_1 + \xi^6_5\xi^2_4\xi^2_1\xi^1_3 - \xi^6_5{\xi^2_3}^2\xi^1_2 + \xi^6_5\xi^2_3\xi^2_2\xi^1_3 - {\xi^5_5}^2\xi^2_4\xi^2_1 + {\xi^5_5}^2\xi^2_1\xi^1_3 - \xi^2_4\xi^2_1 + \xi^2_1\xi^1_3)/({\xi^5_5}^2 + 1)$.
The solution $J$ that's looked for is of type $\xi^2_1 \neq 0$ hence belongs in one of the 3 cases (39), (40), (41). If $J$ belongs in case (40) or (41), the system $S$ is equivalent to the 3 equations (42). If $J$ belongs in the case (39), the system $S$ is equivalent to the 3 equations (42) if and only if $c(J) \neq 0$ where $c(J) = ({\xi^5_5}^2 + 1)\xi^2_1 + \xi^2_1\xi^2_4\xi^6_5 - (\xi^5_5 - \xi^2_2)\xi^2_3\xi^6_5$. Now, if $K$ belongs in case (40), $c(K) = \xi^2_1(K)(\xi^5_5(K)^2 + 1) + \xi^2_4(K)\xi^6_5(K)) = \frac{1}{\xi^1_3(K)}\,\xi^2_1(K)\xi^2_4(K)(\xi^5_5(K)^2 + 1) \neq 0$ since in that case $\xi^6_5(K) = \frac{\xi^5_5(K)^2+1}{\xi^2_4(K)\xi^1_3(K)}\,(\xi^2_4(K) - \xi^1_3(K))$ (see (27)). If $K$ belongs in case (41), $c(K) = \xi^2_1(K)(\xi^5_5(K)^2 + 1) \neq 0$ (see (29)). Hence in both cases, one has $c(J) \neq 0$ in some neighborhood of $K$ and the remaining system is equivalent in that neighborhood to the 3 equations (42). We will now show that the system (42) is of maximal rank 3 at $K$, that is some 3-jacobian doesn't vanish.

- Suppose $K$ belongs in case (40). Then $\frac{D(f,g,h)}{D(\xi^1_1,\xi^1_2,\xi^6_5)}(K) = -\frac{\xi^2_4(K)\xi^2_1(K)^3\xi^1_3(K)}{\xi^5_5(K)^2+1} \neq 0$.
- Suppose $K$ belongs in case (41). Then $\frac{D(f,g,h)}{D(\xi^1_1,\xi^1_2,\xi^2_4)}(K) = -\xi^2_1(K)^3 \neq 0$.

Hence the system (42) is of maximal rank 3 at $K$, and it follows that $\mathfrak{X}_{\xi^2_1\neq 0}$ is is a local submanifold in the neighborhood of $K$.

Hence $\mathfrak{X}_{\xi^2_1\neq 0}$ is is a 12-dimensional submanifold of $\mathbb{R}^{36}$, and so is $\mathfrak{X}_{6,1}$ from (38). Any element of $\mathfrak{X}_{6,1}$ is is equivalent to either $J$ in (31), or $J_\alpha (\alpha \neq 0)$ in (26), or $J''_1$, or $J''_0$ in (37).

## 6.7

$$X_1 = \frac{\partial}{\partial x^1} - y^1\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - x^2\frac{\partial}{\partial y^3}.$$

### 6.7.1 Holomorphic functions for $J_\alpha$.

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J_\alpha$ (26). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}^-_j\ f = 0\ \forall j\ 1 \leqslant j \leqslant 6\}$. As $\tilde{X}^-_2 = -i(\alpha\tilde{X}^-_1 + (1-\alpha)\tilde{X}^-_3)$, $\tilde{X}^-_4 = -i\alpha\tilde{X}^-_1 + (1-i)\tilde{X}^-_3$, $\tilde{X}^-_6 = -i\tilde{X}^-_5$, one has $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}^-_j\ f = 0\ \forall j = 1, 3, 5\}$. Now

$$\begin{aligned} \tilde{X}^-_1 &= 2\frac{\partial}{\partial\overline{z^1}} + i(\alpha - 1)\left(\frac{\partial}{\partial x^2} - \frac{1}{\alpha}\frac{\partial}{\partial y^2}\right) - y^1\frac{\partial}{\partial x^3} - (y^2 + ix^2)\frac{\partial}{\partial y^3} \quad , \\ \tilde{X}^-_3 &= i\frac{\partial}{\partial y^1} + (1 + i\alpha)\frac{\partial}{\partial x^2} - i\frac{\partial}{\partial y^2} - ix^2\frac{\partial}{\partial y^3} \quad , \quad \tilde{X}^-_5 = 2\frac{\partial}{\partial\overline{z^3}} \quad , \end{aligned}$$

where $z^1 = x^1 + iy^1$, $z^3 = x^3 + iy^3$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $z^3$ and satisfies the 2 equations

$$2\frac{\partial f}{\partial\overline{w^2}} + \frac{\partial f}{\partial\overline{z^1}} - \frac{\partial f}{\partial z^1} + \frac{1}{2}\left((1 - i\alpha)w^2 + (1 + i\alpha)\overline{w^2}\right)\frac{\partial f}{\partial z^3} = 0 \tag{43}$$

$$2\frac{\partial f}{\partial\overline{z^1}} + \frac{\alpha - 1}{\alpha}\left(\frac{\partial f}{\partial\overline{w^2}} - \frac{\partial f}{\partial w^2}\right) + \left(\frac{\overline{z^1} - z^1}{2i} + \left(1 - \frac{i\alpha}{2}\right)w^2 + \frac{i\alpha}{2}\overline{w^2}\right)\frac{\partial f}{\partial z^3} = 0 \tag{44}$$

where $w^2 = x^2 + \alpha y^2 - iy^2$. We set $w^1 = z^1, w^3 = z^3$. The 3 functions

$$\varphi^1 = 2w^1 + \overline{w^2} + w^2 \quad , \quad \varphi^2 = 2w^2 + \frac{\alpha - 1}{\alpha}(\overline{w^1} + w^1) \quad ,$$

$$\varphi^3 = w^3 + \frac{1}{32}\left(4i(\overline{w^1})^2 - 8i\overline{w^1 w^2} - 8i\overline{w^1}w^1 - 8(2-i)\overline{w^1}w^2 - (4+i)(\overline{w^2})^2 + 4i\overline{w^2}w^1 + 4i\overline{w^2}w^2\right)$$

if $\alpha = 1$ and if not

$$\varphi^3 = w^3 - \frac{\alpha i}{2(\alpha-1)}\,\overline{w^1}w^2 - \frac{1+i\alpha}{8}\,(\overline{w^2})^2 - \frac{1-\alpha+i\alpha(1+\alpha)}{4(\alpha-1)}\,\overline{w^2}w^2 + \frac{i\alpha}{2(\alpha-1)}w^1w^2 - \frac{3\alpha^2-2\alpha-1-i\alpha(\alpha+1)^2}{8(\alpha-1)^2}\,(w^2)^2$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$w^1_{ax} = w^1_a + w^1_x \tag{45}$$
$$w^2_{ax} = w^2_a + w^2_x \tag{46}$$
$$w^3_{ax} = w^3_a + w^3_x - b^1x^1 - i(b^2x^1 + a^2y^1). \tag{47}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi(a,x) \quad ,$$

where for $\alpha \neq 1$

$$\chi(a,x) = \frac{1}{8}\,\varphi^1_x\left((\overline{\varphi^1_a} + \varphi^1_a)((1-i)\alpha - 1) - \alpha\,\overline{\varphi^2_a} + \frac{\alpha(1-\alpha+2i\alpha)}{\alpha-1}\,\varphi^2_a\right) + \frac{1}{8}\,\varphi^2_x\left(2\frac{i\alpha}{1-i}\,\overline{\varphi^1_a} + \alpha\overline{\varphi^2_a} + 2\frac{\alpha(1+i\alpha)}{(\alpha-1)(1-i)}\,\varphi^1_a + \alpha\frac{(1-2i)\alpha^2-1}{(\alpha-1)^2}\,\varphi^2_a\right)$$

and for $\alpha = 1$

$$\chi(a,x) = \frac{1}{32}\,\varphi^1_x\left(-4i\overline{\varphi^1_a} + 2i\varphi^1_a - 4\overline{\varphi^2_a} + 3i\varphi^2_a\right) + \frac{1}{64}\,\varphi^2_x\left(8(i-1)\,\overline{\varphi^1_a} + 8\overline{\varphi^2_a} - 2i\,\varphi^1_a + (4-5i)\,\varphi^2_a\right).$$

### 6.7.2 Holomorphic functions for $J$.

Now $J$ is defined in (31). Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $z^2$ and $z^3$ and satisfies the equation

$$2\,\frac{\partial f}{\partial \overline{w^1}} = (z^2 + y^1)\,\frac{\partial f}{\partial \overline{z^3}}$$

where $w^1 = x^1 - iy^1$, $z^2 = x^2 + iy^2$, $z^3 = x^3 + iy^3$. We set $w^2 = z^2, w^3 = z^3$. The 3 functions

$$\varphi^1 = w^1 \quad , \quad \varphi^2 = w^2 \quad , \quad \varphi^3 = w^3 + \frac{1}{2}w^2\overline{w^1} + \frac{i}{4}w^1\overline{w^1} - \frac{i}{8}\overline{w^1}^2$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$, and

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi(a,x) \quad ,$$

where from (47)

$$\chi(a,x) = \frac{1}{4}\,\varphi^1_x\left(2i\overline{\varphi^1_a} - i\varphi^1_a + 2\overline{\varphi^2_a}\right) + \frac{1}{2}\,\varphi^2_x\overline{\varphi^1_a}.$$

### 6.7.3 Holomorphic functions for $J''_\gamma$.

$J''_\gamma$ is defined in (37) for any real $\gamma$. Here $\tilde{X}_2^- = i\tilde{X}_1^-$, $\tilde{X}_4^- = -i\tilde{X}_2^- - i\gamma\tilde{X}_1^-$, $\tilde{X}_6^- = -\frac{i}{2}\,\tilde{X}_5^-$, hence $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^-\, f = 0\ \forall j = 1, 2, 5\}$. One has

$$\begin{aligned}
\tilde{X}_1^- &= 2\,\frac{\partial}{\partial \overline{w^1}} - i\frac{\partial}{\partial x^2} - y^1\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3} \quad , \\
\tilde{X}_2^- &= 2\,\frac{\partial}{\partial \overline{w^2}} + i\gamma\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial y^3} \quad , \quad \tilde{X}_5^- = 2\,\frac{\partial}{\partial \overline{w^3}} \quad ,
\end{aligned}$$

where $w^1 = x^1 - ix^2 \quad , \quad w^1 = y^1 + iy^2 \quad , \quad w^3 = x^3 + \frac{i}{2}y^3$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^3$ and satisfies the 2 equations

$$2\,\frac{\partial f}{\partial \overline{w^2}} - \frac{\gamma}{2}\left(\frac{\partial f}{\partial \overline{w^1}} - \frac{\partial f}{\partial w^1}\right) + \frac{1}{4}\left(w^1 - \overline{w^1}\right)\frac{\partial f}{\partial w^3} = 0 \tag{48}$$

$$2\,\frac{\partial f}{\partial \overline{w^1}} - \frac{1}{4}\left(3w^2 + \overline{w^2}\right)\frac{\partial f}{\partial w^3} = 0\,. \tag{49}$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= \gamma\overline{w^2} - 4w^1 \quad , \quad \varphi^2 = w^2 \quad , \\
\varphi^3 &= w^3 + \frac{1}{32}\left(4\overline{w^1 w^2} + 12\overline{w^1}w^2 + \gamma(\overline{w^2})^2 - 4\overline{w^2}w^1 + 12w^1w^2\right)
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. Instead of (47), we here have

$$w^3_{ax} = w^3_a + w^3_x - b^1x^1 - \frac{i}{2}\,(b^2x^1 + a^2y^1),$$

whence

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi(a, x)$$

with

$$\chi(a, x) = \frac{1}{16}\,\varphi^1_x\overline{\varphi^2_a} + \frac{1}{16}\,\varphi^2_x\left(-\overline{\varphi^1_a} - 2\varphi^1_a + 2\gamma\overline{\varphi^2_a} + \gamma\varphi^2_a\right).$$

# 7 Lie Algebra $\mathcal{G}_{6,6}$ (isomorphic to $M1$).

Commutation relations for $\mathcal{G}_{6,6}$ : $[x_1, x_2] = x_4$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_5$.

$$J = \begin{pmatrix}
\boxed{\xi_1^1} & -\frac{{\xi_1^1}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\
\boxed{\xi_1^2} & -\xi_1^1 & 0 & 0 & 0 & 0 \\
* & * & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & 0 & 0 \\
\boxed{\xi_1^4} & \boxed{\xi_2^4} & \boxed{\xi_3^4} & -\xi_3^3 & 0 & 0 \\
\boxed{\xi_1^5} & * & \boxed{\xi_3^5} & \boxed{\xi_4^5} & -\xi_3^3 & \xi_3^4 \\
\boxed{\xi_1^6} & * & -\xi_4^5 & * & -\frac{{\xi_3^3}^2+1}{\xi_3^4} & \xi_3^3
\end{pmatrix} \tag{50}$$

where $\mathbf{J_1^3} = ((\xi_3^3 - \xi_1^1)\xi_1^4 - \xi_2^4\xi_1^2)/\xi_3^4$; $\mathbf{J_2^3} = ((\xi_3^3 + \xi_1^1)\xi_2^4\xi_1^2 + ({\xi_1^1}^2 + 1)\xi_1^4)/(\xi_3^4\xi_1^2)$; $\mathbf{J_2^5} = (-((\xi_3^3 - \xi_1^1)\xi_3^5\xi_1^4 - (\xi_3^3 - \xi_1^1)\xi_1^5\xi_3^4 - \xi_3^5\xi_2^4\xi_1^2 + \xi_4^5\xi_3^4\xi_1^4 + \xi_1^6{\xi_3^4}^2))/(\xi_3^4\xi_1^2)$; $\mathbf{J_2^6} = (-((((\xi_3^3 + \xi_1^1)\xi_1^4 + \xi_2^4\xi_1^2)\xi_4^5\xi_3^4 + (\xi_3^5\xi_1^4 - \xi_1^5\xi_3^4)({\xi_3^3}^2 + 1) + (\xi_3^3 + \xi_1^1)\xi_1^6{\xi_3^4}^2))/({\xi_3^4}^2\xi_1^2)$; $\mathbf{J_4^6} = (({\xi_3^3}^2 + 1)\xi_3^5 + 2\xi_4^5\xi_3^4\xi_3^3)/{\xi_3^4}^2$; and the parameters are subject to the condition

$$\xi_1^2\xi_3^4 \neq 0. \tag{51}$$

Now the automorphism group of $\mathcal{G}_{6,6}$ is comprised of the matrices

$$\Phi = \begin{pmatrix}
b_1^1 & b_2^1 & 0 & 0 & 0 & 0 \\
0 & b_2^2 & b_3^2 & 0 & 0 & 0 \\
b_1^3 & b_2^3 & b_3^3 & 0 & 0 & 0 \\
b_1^4 & b_2^4 & b_3^4 & b_2^2b_1^1 & 0 & 0 \\
b_1^5 & b_2^5 & b_3^5 & -b_1^4b_2^2 & {b_2^2}^2b_1^1 & b_3^4b_2^2 \\
b_1^6 & b_2^6 & b_3^6 & -b_1^3b_2^2 & 0 & b_3^3b_2^2
\end{pmatrix}$$

where $b_1^1 b_2^2 b_3^3 \neq 0$. Taking

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(\xi_4^5 \xi_3^4)/({\xi_3^3}^2+1) & 0 & 1 & 0 \\ b_1^6 & b_2^6 & -(({\xi_3^3}^2+1)\xi_3^5 + 2\xi_4^5\xi_3^4\xi_3^3)/(({\xi_3^3}^2+1)\xi_3^4) & 0 & 0 & 1 \end{pmatrix}$$

with suitable values for $b_1^6, b_2^6$, equivalence by $\Phi$ leads to the case where $\xi_1^5 = \xi_3^5 = \xi_4^5 = \xi_1^6 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & (-\xi_2^4\xi_1^2 + \xi_1^4\xi_3^3 - \xi_1^4\xi_1^1)/(\xi_3^4\xi_1^2) & 1 & 0 & 0 & 0 \\ 0 & \xi_1^4/\xi_1^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_1^4 = \xi_2^4 = 0$. Finally equivalence by $\Phi = \operatorname{diag}\left(\begin{pmatrix} \xi_1^2 & -\xi_1^1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} (\xi_3^4\xi_1^2)/({\xi_3^3}^2+1) & 0 \\ -(\xi_3^4\xi_3^3\xi_1^2)/({\xi_3^3}^2+1) & \xi_1^2 \end{pmatrix}, \begin{pmatrix} \xi_1^2 & -(\xi_3^4\xi_3^3\xi_1^2)/({\xi_3^3}^2+1) \\ 0 & (\xi_3^4\xi_1^2)/({\xi_3^3}^2+1) \end{pmatrix}\right)$ leads to

$$J = \operatorname{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right). \tag{52}$$

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = -\tilde{x}_5$; $[\tilde{x}_1, \tilde{x}_4] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = \tilde{x}_5$.

From (50), $\mathfrak{X}_{6,6}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$. There is only one Aut $\mathcal{G}_{6,6}$ orbit, and any element of $\mathfrak{X}_{6,6}$ is equivalent to $J$ in (52).

## 7.1

$$X_1 = \frac{\partial}{\partial x^1} - y^1 \frac{\partial}{\partial y^2} + \frac{(y^1)^2}{2} \frac{\partial}{\partial x^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - y^2 \frac{\partial}{\partial x^3} - x^2 \frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J$ in (52). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0) \; ; \; \tilde{X}_j^- f = 0 \; \forall j = 1,3,5\}$. One has

$$\begin{aligned} \tilde{X}_1^- &= 2 \frac{\partial}{\partial \overline{w^1}} - y^1 \frac{\partial}{\partial y^2} + \left( \frac{(y^1)^2}{2} - iy^2 \right) \frac{\partial}{\partial x^3} - ix^2 \frac{\partial}{\partial y^3}, \\ \tilde{X}_3^- &= 2 \frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}_5^- = 2 \frac{\partial}{\partial \overline{w^3}} \quad , \end{aligned}$$

where $w^1 = x^1 + iy^1 \quad , \quad w^2 = x^2 + iy^2 \quad , \quad w^3 = x^3 - iy^3$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$2 \frac{\partial f}{\partial \overline{w^1}} - \frac{w^1 - \overline{w^1}}{2} \frac{\partial f}{\partial w^2} - \left( \frac{(w^1 - \overline{w^1})^2}{8} + w^2 \right) \frac{\partial f}{\partial w^3} = 0. \tag{53}$$

The 3 functions

$$\begin{aligned} \varphi^1 &= w^1 \quad , \quad \varphi^2 = w^2 + \frac{1}{4} \left( w^1 \overline{w^1} - \frac{(\overline{w^1})^2}{2} \right) \quad , \\ \varphi^3 &= w^3 + \frac{1}{48} \left( -(\overline{w^1})^3 + 3\overline{w^1}(w^1)^2 + 24\overline{w^1} w^2 \right) \end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective

second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$w^1_{ax} = w^1_a + w^1_x \quad , \quad w^2_{ax} = w^2_a + w^2_x - ib^1x^1 \quad , \quad w^3_{ax} = w^3_a + w^3_x + \frac{(b^1)^2}{2}x^1 - (b^2 - b^1x^1)y^1 + ia^2y^1.$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x + \frac{1}{4}\,\varphi^1_x(2\overline{\varphi^1_a} - \varphi^1_a) \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi(a,x) \quad ,$$

where

$$\chi(a,x) = \frac{1}{16}\,(\varphi^1_x)^2\left(3\overline{\varphi^1_a} - 2\varphi^1_a\right) + \frac{1}{16}\,\varphi^1_x\left(2(\overline{\varphi^1_a})^2 - (\varphi^1_a)^2 + 8\overline{\varphi^2_a}\right) + \frac{1}{2}\,\varphi^2_x\overline{\varphi^1_a}.$$

# 8 Lie Algebra $\mathcal{G}_{6,5}$ (isomorphic to $M8$).

Commutation relations for $\mathcal{G}_{6,5}$ : $[x_1, x_2] = x_4$; $[x_1, x_4] = x_5$; $[x_2, x_3] = x_6$; $[x_2, x_4] = x_6$.

$$J = \begin{pmatrix} a & -\frac{a^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & -a & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & * & b & -\frac{b^2+1}{\xi_3^4} & 0 & 0 \\ \boxed{\xi_1^4} & * & \boxed{\xi_3^4} & -b & 0 & 0 \\ \boxed{\xi_1^5} & * & \frac{((2\xi_5^5+\xi_3^4)\xi_3^6-\xi_4^6\xi_3^4)\xi_1^2-\xi_5^6\xi_3^6\xi_3^4}{\xi_5^6\xi_1^2} & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & * & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{54}$$

where $\mathbf{a} = \mathbf{J_1^1} = (-(({\xi_5^5}^2 + 1)\xi_1^2 - \xi_5^6\xi_5^5\xi_3^4))/(\xi_5^6\xi_3^4)$; $\mathbf{J_2^3} = ((((\xi_5^5 + 2\xi_3^4)\xi_5^5\xi_1^4 + {\xi_3^4}^2\xi_1^4 + {\xi_3^4}^2\xi_1^3 + \xi_1^4)\xi_5^6 + ({\xi_5^5}^2 + 1)\xi_1^3\xi_1^2){\xi_1^2}^2 + ((\xi_5^6\xi_3^4 - 2\xi_5^5\xi_1^2)\xi_1^4 - (2\xi_1^4 + \xi_1^3)\xi_3^4\xi_1^2){\xi_5^6}^2\xi_3^4)/(\xi_5^6\xi_3^4{\xi_1^2}^3)$; $\mathbf{b} = \mathbf{J_3^3} = (-((\xi_5^5 + \xi_3^4)\xi_1^2 - \xi_5^6\xi_3^4))/\xi_1^2$; $\mathbf{J_2^4} = ((({\xi_5^5}^2 + 1){\xi_1^2}^2 + {\xi_5^6}^2{\xi_3^4}^2)\xi_1^4 - ((\xi_1^4 + \xi_1^3)\xi_3^4 + 2\xi_5^5\xi_1^4)\xi_5^6\xi_3^4\xi_1^2)/(\xi_5^6\xi_3^4{\xi_1^2}^2)$; $\mathbf{J_2^5} = (-(((((\xi_5^5\xi_1^4 + 2\xi_3^4\xi_1^4 + 2\xi_3^4\xi_1^3)\xi_5^5 + {\xi_3^4}^2\xi_1^4 + {\xi_3^4}^2\xi_1^3 + \xi_1^4)\xi_3^6 - (\xi_1^6\xi_3^4 + \xi_1^5\xi_1^2)({\xi_5^5}^2 + 1) - (\xi_1^4 + \xi_1^3)\xi_4^6{\xi_3^4}^2){\xi_1^2}^2 - ((2(\xi_3^6\xi_1^4 - \xi_1^5\xi_1^2)\xi_5^5 + (2\xi_1^4 + \xi_1^3)\xi_3^6\xi_3^4)\xi_1^2 - (\xi_5^6\xi_3^6 + \xi_4^6\xi_1^2)\xi_3^4\xi_1^4)\xi_5^6\xi_3^4))/(\xi_5^6\xi_3^4{\xi_1^2}^3)$; $\mathbf{J_4^5} = (-(((\xi_4^6{\xi_3^4}^2 - \xi_3^6)\xi_1^2 - \xi_5^6\xi_4^6{\xi_3^4}^2)\xi_1^2 - ((\xi_5^5 + \xi_3^4)\xi_1^2 - \xi_5^6\xi_3^4)^2\xi_3^6))/(\xi_5^6\xi_3^4{\xi_1^2}^2)$; $\mathbf{J_2^6} = (-((\xi_5^6\xi_1^5 + \xi_4^6\xi_1^4 + \xi_3^6\xi_1^3)\xi_5^6\xi_3^4 - ({\xi_5^5}^2 + 1)\xi_1^6\xi_1^2))/(\xi_5^6\xi_3^4\xi_1^2)$; and the parameters are subject to the condition

$$\xi_1^2\xi_3^4\xi_5^6 \neq 0. \tag{55}$$

The automorphisms of $\mathcal{G}_{6,5}$ fall into 2 kinds The first kind is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_2^2 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_2^2b_1^1 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & b_2^2b_1^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_2^4b_1^1 & b_2^2{b_1^1}^2 & 0 \\ b_1^6 & b_2^6 & b_3^6 & -(b_1^4 + b_1^3)b_2^2 & 0 & {b_2^2}^2b_1^1 \end{pmatrix} \tag{56}$$

where $b_1^1b_2^2 \neq 0$. The second kind is comprised of the matrices

$$\Phi = \begin{pmatrix} 0 & b_2^1 & 0 & 0 & 0 & 0 \\ b_1^2 & 0 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_1^2b_2^1 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & -b_1^2b_2^1 & -b_1^2b_2^1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & -b_1^4b_2^1 & 0 & -b_1^2{b_2^1}^2 \\ b_1^6 & b_2^6 & b_3^6 & b_1^2(b_2^4 + b_2^3) & -{b_1^2}^2b_2^1 & 0 \end{pmatrix} \tag{57}$$

where $b_2^1b_1^2 \neq 0$. Taking suitable values for the $b_j^i$'s, equivalence by $\Phi$ in (56) leads to the case where

$\xi_1^5=\xi_5^5=\xi_1^6=\xi_3^6=\xi_4^6=\xi_5^6=\xi_1^3=\xi_1^4=0$ and moreover $\xi_1^2=1$ :

$$J(\xi_3^4,\xi_5^5,\xi_5^6)=\begin{pmatrix} -\frac{{\xi_5^5}^2+1-\xi_5^6\xi_5^5\xi_3^4}{\xi_5^6\xi_3^4} & -\frac{({\xi_5^5}^2+1-\xi_5^6\xi_5^5\xi_3^4)^2}{(\xi_5^6\xi_3^4)^2}-1 & 0 & 0 & 0 & 0 \\ 1 & \frac{{\xi_5^5}^2+1-\xi_5^6\xi_5^5\xi_3^4}{\xi_5^6\xi_3^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\xi_5^5+\xi_3^4-\xi_5^6\xi_3^4) & -\frac{(\xi_5^5+\xi_3^4-\xi_5^6\xi_3^4)^2+1}{\xi_3^4} & 0 & 0 \\ 0 & 0 & \xi_3^4 & \xi_5^5+\xi_3^4-\xi_5^6\xi_3^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ 0 & 0 & 0 & 0 & \xi_5^6 & -\xi_5^5 \end{pmatrix} \tag{58}$$

where $\xi_3^4\xi_5^6\neq 0$.

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1,\tilde{x}_3]=\frac{1}{\xi_5^6}\left((-\xi_5^5\xi_3^4+{\xi_5^5}^2+1)\tilde{x}_5+(-\xi_5^6\xi_3^4+\xi_5^5)\tilde{x}_6\right)$;
$[\tilde{x}_1,\tilde{x}_4]=\frac{1}{\xi_5^6\xi_3^4}\left(({\xi_5^5}^2+1)(\xi_5^5+\xi_3^4)+\xi_5^6\xi_5^5\xi_3^4(\xi_5^6\xi_3^4-2\xi_5^5-\xi_3^4)\right)\tilde{x}_5+\frac{1}{\xi_3^4}\left((\xi_5^6\xi_3^4-\xi_5^5)^2+\xi_3^4(\xi_5^5-\xi_5^6\xi_3^4)+1\right)\tilde{x}_6$;
$[\tilde{x}_2,\tilde{x}_3]=\frac{(\xi_5^6\xi_3^4-\xi_5^5)^2+1}{{\xi_5^6}^2\xi_3^4}\left((1+{\xi_5^5}^2)\tilde{x}_5+\xi_5^6\xi_5^5\tilde{x}_6\right)$;
$[\tilde{x}_2,\tilde{x}_4]=\frac{(\xi_5^6\xi_3^4-\xi_5^5)^2+1}{{\xi_5^6}^2{\xi_3^4}^2}\left(-({\xi_5^5}^2+1)(\xi_3^4\xi_5^6-\xi_5^5-\xi_3^4)\tilde{x}_5+\xi_5^6(-\xi_5^6\xi_5^5\xi_3^4+{\xi_5^5}^2+1+\xi_5^5\xi_3^4)\tilde{x}_6\right)$.

$J(\xi_3^4,\xi_5^5,\xi_5^6),J(\eta_3^4,\eta_5^5,\eta_5^6)$ as in (58) are equivalent under some first kind automorphism if and only if $\eta_3^4=\xi_3^4,\eta_5^5=\xi_5^5,\eta_5^6=\xi_5^6$. They are equivalent under some second kind automorphism if and only if $\eta_3^4=-((\xi_5^6\xi_3^4-\xi_5^5)^2+1)/\xi_3^4$, $\eta_5^5=-\xi_5^5$, $\eta_5^6=\xi_5^6{\xi_3^4}^2/((\xi_5^6\xi_3^4-\xi_5^5)^2+1)$.

From (54), $\mathfrak{X}_{6,5}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. Each CS is equivalent to some $J(\xi_3^4,\xi_5^5,\xi_5^6)$ in (58).

## 8.1

$$X_1=\frac{\partial}{\partial x^1}-y^1\frac{\partial}{\partial y^2}-y^2\frac{\partial}{\partial x^3}+\frac{(y^1)^2}{2}\frac{\partial}{\partial y^3}\quad,\quad X_2=\frac{\partial}{\partial y^1}-(x^2+y^2)\frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J(\xi_3^4,\xi_5^5,\xi_5^6)$ in (58) where $\xi_3^4\xi_5^6\neq 0$. Then $H_{\mathbb{C}}(G)=\{f\in C^\infty(G_0)\ ;\ \tilde{X}_j^-\,f=0\ \forall j=1,3,5\}$. One has

$$\begin{aligned}\tilde{X}_1^- &= 2\frac{\partial}{\partial\overline{w^1}}-y^1(1+iA)\frac{\partial}{\partial y^2}-y^2(1+iA)\frac{\partial}{\partial x^3}+\left(\frac{(y^1)^2}{2}(1+iA)-i(x^2+y^2)\right)\frac{\partial}{\partial y^3}\quad,\\ \tilde{X}_3^- &= 2\frac{\partial}{\partial\overline{w^2}}\quad,\quad \tilde{X}_5^-=2\frac{\partial}{\partial\overline{w^3}}\quad,\end{aligned}$$

where

$$\begin{aligned}w^1 &= x^1-Ay^1+iy^1\\ w^2 &= x^2+\frac{\xi_5^5+\xi_3^4-\xi_5^6\xi_3^4}{\xi_3^4}\,y^2+\frac{i}{\xi_3^4}\,y^2\\ w^3 &= x^3-\frac{\xi_5^5}{\xi_5^6}\,y^3+\frac{i}{\xi_5^6}\,y^3\\ A &= -\frac{{\xi_5^5}^2+1-\xi_5^6\xi_5^5\xi_3^4}{\xi_5^6\xi_3^4}.\end{aligned}$$

Then $f\in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$\begin{aligned}&2\frac{\partial f}{\partial\overline{w^1}}-\frac{w^1-\overline{w^1}}{2i}\,\frac{(1+iA)(\xi_5^5+\xi_3^4-\xi_5^6\xi_3^4+i)}{\xi_3^4}\,\frac{\partial f}{\partial w^2}\\ &-\left[(1+iA)\xi_3^4\,\frac{w^2-\overline{w^2}}{2i}+\left(\frac{(w^1-\overline{w^1})^2}{8}(1+iA)+i\left(\frac{w^2+\overline{w^2}}{2}-(\xi_5^5-\xi_5^6\xi_3^4)\,\frac{w^2-\overline{w^2}}{2i}\right)\right)\frac{i-\xi_5^5}{\xi_5^6}\right]\frac{\partial f}{\partial w^3}=0.\end{aligned}$$

The 3 functions

$$\varphi^1 = w^1 \quad , \quad \varphi^2 = w^2 + \frac{1+iA}{4i\xi_3^4}(\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4 + i)\left( w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2} \right) \quad ,$$

$$\begin{aligned}
\varphi^3 = w^3 &+ \frac{1}{48{\xi_5^6}^2{\xi_3^4}^2}\,\overline{w^1}^3(-2i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 - 4{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + 2i{\xi_5^6}^2{\xi_3^4}^2 + 4i\xi_5^6{\xi_5^5}^3\xi_3^4 + i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 + 4\xi_5^6{\xi_5^5}^2\xi_3^4 + 2\xi_5^6\xi_5^5{\xi_3^4}^2 \\
&+ 4i\xi_5^6\xi_5^5\xi_3^4 - i\xi_5^6{\xi_3^4}^2 + 4\xi_5^6\xi_3^4 - 2i{\xi_5^5}^4 - i{\xi_5^5}^3\xi_3^4 - {\xi_5^5}^2\xi_3^4 - 4i{\xi_5^5}^2 - i\xi_5^5\xi_3^4 - \xi_3^4 - 2i) \\
&+ \frac{1}{16{\xi_5^6}^2{\xi_3^4}^2}\,\overline{w^1}^2 w^1(i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 + 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 - 2i\xi_5^6{\xi_5^5}^3\xi_3^4 - 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2i\xi_5^6\xi_5^5\xi_3^4 - 2\xi_5^6\xi_3^4 + i{\xi_5^5}^4 + 2i{\xi_5^5}^2 + i) \\
&+ \frac{1}{16{\xi_5^6}^2\xi_3^4}\,\overline{w^1}{w^1}^2(-i\xi_5^6{\xi_5^5}^2\xi_3^4 - 2\xi_5^6\xi_5^5\xi_3^4 + i\xi_5^6\xi_3^4 + i{\xi_5^5}^3 + {\xi_5^5}^2 + i\xi_5^5 + 1) - \frac{i\xi_5^5 + 1}{2\xi_5^6}\,\overline{w^1}w^2
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned}
w^1_{ax} &= w^1_a + w^1_x \\
w^2_{ax} &= w^2_a + w^2_x - \frac{b^1x^1}{\xi_3^4}(\xi_5^5 + \xi_3^4 - \xi_5^6\xi_3^4 + i) \\
w^3_{ax} &= w^3_a + w^3_x - b^2x^1 + \frac{1}{2}b^1(x^1)^2 + \frac{i - \xi_5^5}{\xi_5^6}\left( \frac{1}{2}(b^1)^2x^1 - b^2y^1 + b^1x^1y^1 - a^2y^1 \right).
\end{aligned}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x + C\varphi^1_x \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + D_1(\varphi^1_x)^2 + D_2\varphi^1_x + D_3\varphi^2_x$$

where $C = (\overline{\varphi^1_a}(i\xi_5^6\xi_3^4 - i\xi_5^5 - i\xi_3^4 + 1))/(2\xi_3^4) + (\varphi^1_a(-{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^4 + \xi_5^6\xi_5^5{\xi_3^4}^2 + 2i\xi_5^6\xi_5^5\xi_3^4 + i\xi_5^6{\xi_3^4}^2 - {\xi_5^5}^3 - {\xi_5^5}^2\xi_3^4 - i{\xi_5^5}^2 - \xi_5^5 - \xi_3^4 - i))/(4\xi_5^6{\xi_3^4}^2)$;
$D_1 = (\overline{\varphi^1_a}(-i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 + 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + i{\xi_5^6}^2{\xi_3^4}^2 + 2i\xi_5^6{\xi_5^5}^3\xi_3^4 + i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 - 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2\xi_5^6\xi_5^5{\xi_3^4}^2 + 2i\xi_5^6\xi_5^5\xi_3^4 + 3i\xi_5^6{\xi_3^4}^2 - 2\xi_5^6\xi_3^4 - i{\xi_5^5}^4 - i{\xi_5^5}^3\xi_3^4 - {\xi_5^5}^2\xi_3^4 - 2i{\xi_5^5}^2 - i\xi_5^5\xi_3^4 - \xi_3^4 - i))/(16{\xi_5^6}^2{\xi_3^4}^2) + (\varphi^1_a(i{\xi_5^6}^2{\xi_5^5}^2{\xi_3^4}^2 - 2{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - i{\xi_5^6}^2{\xi_3^4}^2 - 2i\xi_5^6{\xi_5^5}^3\xi_3^4 - 2i\xi_5^6{\xi_5^5}^2{\xi_3^4}^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^4 - 2i\xi_5^6\xi_5^5\xi_3^4 - 2i\xi_5^6{\xi_3^4}^2 + 2\xi_5^6\xi_3^4 + i{\xi_5^5}^4 + 2i{\xi_5^5}^3\xi_3^4 + 2{\xi_5^5}^2\xi_3^4 + 2i{\xi_5^5}^2 + 2i\xi_5^5\xi_3^4 + 2\xi_3^4 + i))/(16{\xi_5^6}^2{\xi_3^4}^2)$;
$D_2 = (\overline{\varphi^1_a}^2(-i{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 - {\xi_5^6}^2{\xi_3^4}^2 + 2i\xi_5^6{\xi_5^5}^2\xi_3^4 + i\xi_5^6\xi_5^5{\xi_3^4}^2 + 4\xi_5^6\xi_5^5\xi_3^4 + \xi_5^6{\xi_3^4}^2 - 2i\xi_5^6\xi_3^4 - i{\xi_5^5}^3 - i{\xi_5^5}^2\xi_3^4 - 3{\xi_5^5}^2 - 2\xi_5^5\xi_3^4 - i\xi_5^5 + i\xi_3^4 - 3))/(16\xi_5^6\xi_3^4) + (\overline{\varphi^1_a}\varphi^1_a(\xi_5^6\xi_3^4 - \xi_5^5 - \xi_3^4 + i))/4 + (-i\overline{\varphi^2_a}\xi_3^4)/2 + ({\varphi^1_a}^2(i{\xi_5^6}^3\xi_5^5{\xi_3^4}^2 - {\xi_5^6}^3{\xi_3^4}^2 - 2i{\xi_5^6}^2{\xi_5^5}^2\xi_3^4 - i{\xi_5^6}^2\xi_5^5{\xi_3^4}^2 + {\xi_5^6}^2{\xi_3^4}^2 - 2i{\xi_5^6}^2\xi_3^4 + i\xi_5^6{\xi_5^5}^3 + \xi_5^6{\xi_5^5}^2 + i\xi_5^6\xi_5^5 + \xi_5^6 + i{\xi_5^5}^3 + {\xi_5^5}^2 + i\xi_5^5 + 1))/(16{\xi_5^6}^2\xi_3^4) + (\varphi^2_a(i\xi_5^6\xi_3^4 - i\xi_5^5 - 1))/(2\xi_5^6)$;
$D_3 = (\overline{\varphi^1_a}(-i\xi_5^5 - 1))/(2\xi_5^6)$.

# 9 Lie Algebra $\mathcal{G}_{6,8}$ (isomorphic to $M9$).

Commutation relations for $\mathcal{G}_{6,8}$ : $[x_1, x_2] = x_4$; $[x_1, x_4] = x_5$; $[x_2, x_3] = x_5$; $[x_2, x_4] = x_6$.

$$J = \begin{pmatrix}
\frac{\xi_5^5\xi_3^4 + \xi_5^5\xi_1^2 - \xi_3^3\xi_1^2}{\xi_3^4} & -\frac{(\xi_5^5\xi_3^4 + \xi_5^5\xi_1^2 - \xi_3^3\xi_1^2)^2 + {\xi_3^4}^2}{{\xi_3^4}^2\xi_1^2} & 0 & 0 & 0 & 0 \\
\boxed{\xi_1^2} & -\frac{\xi_5^5\xi_3^4 + \xi_5^5\xi_1^2 - \xi_3^3\xi_1^2}{\xi_3^4} & 0 & 0 & 0 & 0 \\
\boxed{\xi_1^3} & \boxed{\xi_2^3} & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2 + 1}{\xi_3^4} & 0 & 0 \\
* & * & \boxed{\xi_3^4} & -\xi_3^3 & 0 & 0 \\
* & * & -\frac{(\xi_4^6\xi_3^4 - \xi_3^6\xi_5^5 + \xi_3^6\xi_3^3)(\xi_3^4 + \xi_1^2)}{\xi_3^4\xi_1^2} & * & \boxed{\xi_5^5} & -\frac{({\xi_5^5}^2 + 1)(\xi_3^4 + \xi_1^2)}{\xi_3^4\xi_1^2} \\
\boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \frac{\xi_3^4\xi_1^2}{\xi_3^4 + \xi_1^2} & -\xi_5^5
\end{pmatrix} \tag{59}$$

where $\mathbf{J^4_1} = (\xi^4_3\xi^3_3\xi^3_1 + \xi^4_3\xi^3_2\xi^2_1 - \xi^3_3\xi^3_1\xi^2_1 + (\xi^4_3 + \xi^2_1)\xi^5_5\xi^3_1)/({\xi^3_3}^2 + 1)$; $\mathbf{J^4_2} = (-(((\xi^5_5 - \xi^3_3)(\xi^4_3\xi^3_3\xi^3_1 + \xi^4_3\xi^3_2\xi^2_1 - \xi^3_3\xi^3_1\xi^2_1) - (\xi^4_3 + \xi^2_1)\xi^5_5\xi^3_3\xi^3_1)(\xi^4_3 + \xi^2_1) + ((\xi^4_3 + \xi^2_1)^2{\xi^5_5}^2 + ({\xi^3_3}^2 + 1){\xi^4_3}^2)\xi^3_1))/(({\xi^3_3}^2 + 1)\xi^4_3\xi^2_1)$; $\mathbf{J^5_1} = (-(\xi^6_4\xi^5_5{\xi^4_3}^2\xi^3_1 + \xi^6_4\xi^5_5\xi^4_3\xi^3_1\xi^2_1 + \xi^6_4{\xi^4_3}^2\xi^3_3\xi^3_1 + \xi^6_4{\xi^4_3}^2\xi^3_2\xi^2_1 - \xi^6_4\xi^4_3\xi^3_3\xi^3_1\xi^2_1 + \xi^6_3\xi^4_3{\xi^3_3}^2\xi^3_1 + \xi^6_3\xi^4_3\xi^3_1 + \xi^6_2\xi^4_3{\xi^3_3}^2\xi^2_1 + \xi^6_2\xi^4_3\xi^2_1 + \xi^6_1\xi^5_5{\xi^3_3}^2\xi^2_1 + \xi^6_1\xi^5_5\xi^2_1 - \xi^6_1{\xi^3_3}^3\xi^2_1 - \xi^6_1\xi^3_3\xi^2_1)(\xi^4_3 + \xi^2_1))/(({\xi^3_3}^2 + 1){\xi^4_3}^2\xi^2_1)$; $\mathbf{J^5_2} = ((\xi^6_4{\xi^5_5}^2{\xi^4_3}^3\xi^3_1 + 2\xi^6_4{\xi^5_5}^2{\xi^4_3}^2\xi^3_1\xi^2_1 + \xi^6_4{\xi^5_5}^2\xi^4_3\xi^3_1{\xi^2_1}^2 + \xi^6_4\xi^5_5{\xi^4_3}^3\xi^3_2\xi^2_1 - 2\xi^6_4\xi^5_5{\xi^4_3}^2\xi^3_3\xi^3_1\xi^2_1 + \xi^6_4\xi^5_5{\xi^4_3}^2\xi^3_2{\xi^2_1}^2 - 2\xi^6_4\xi^5_5\xi^4_3\xi^3_3\xi^3_1{\xi^2_1}^2 - \xi^6_4{\xi^4_3}^3\xi^3_3\xi^3_2\xi^2_1 + \xi^6_4{\xi^4_3}^3\xi^3_1 - \xi^6_4{\xi^4_3}^2\xi^3_3\xi^3_2{\xi^2_1}^2 + \xi^6_4\xi^4_3{\xi^3_3}^2\xi^3_1{\xi^2_1}^2 - \xi^6_3{\xi^4_3}^2{\xi^3_3}^2\xi^3_2\xi^2_1 - \xi^6_3{\xi^4_3}^2\xi^3_2\xi^2_1 + 2\xi^6_2\xi^5_5{\xi^4_3}^2{\xi^3_3}^2\xi^2_1 + 2\xi^6_2\xi^5_5{\xi^4_3}^2\xi^2_1 + \xi^6_2\xi^5_5\xi^4_3{\xi^3_3}^2{\xi^2_1}^2 + \xi^6_2\xi^5_5\xi^4_3{\xi^2_1}^2 - \xi^6_2\xi^4_3{\xi^3_3}^3{\xi^2_1}^2 - \xi^6_2\xi^4_3\xi^3_3{\xi^2_1}^2 + \xi^6_1{\xi^5_5}^2{\xi^4_3}^2{\xi^3_3}^2 + \xi^6_1{\xi^5_5}^2{\xi^4_3}^2 + 2\xi^6_1{\xi^5_5}^2\xi^4_3{\xi^3_3}^2\xi^2_1 + 2\xi^6_1{\xi^5_5}^2\xi^4_3\xi^2_1 + \xi^6_1{\xi^5_5}^2{\xi^3_3}^2{\xi^2_1}^2 + \xi^6_1{\xi^5_5}^2{\xi^2_1}^2 - 2\xi^6_1\xi^5_5\xi^4_3{\xi^3_3}^3\xi^2_1 - 2\xi^6_1\xi^5_5\xi^4_3\xi^3_3\xi^2_1 - 2\xi^6_1\xi^5_5{\xi^3_3}^3{\xi^2_1}^2 - 2\xi^6_1\xi^5_5\xi^3_3{\xi^2_1}^2 + \xi^6_1{\xi^4_3}^2{\xi^3_3}^2 + \xi^6_1{\xi^4_3}^2 + \xi^6_1{\xi^3_3}^4{\xi^2_1}^2 + \xi^6_1{\xi^3_3}^2{\xi^2_1}^2)(\xi^4_3 + \xi^2_1))/(({\xi^3_3}^2 + 1){\xi^4_3}^3{\xi^2_1}^2)$; $\mathbf{J^5_4} = ((\xi^6_4\xi^5_5\xi^4_3 + \xi^6_4\xi^4_3\xi^3_3 + \xi^6_3{\xi^3_3}^2 + \xi^6_3)(\xi^4_3 + \xi^2_1))/({\xi^4_3}^2\xi^2_1)$; and the parameters are subject to the condition

$$\xi^2_1\xi^4_3(\xi^4_3 + \xi^2_1) \neq 0. \tag{60}$$

Now the automorphism group of $\mathcal{G}_{6,8}$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b^1_1 & b^1_2 & 0 & 0 & 0 & 0 \\ 0 & b^2_2 & 0 & 0 & 0 & 0 \\ b^3_1 & b^3_2 & {b^1_1}^2 & 0 & 0 & 0 \\ b^4_1 & b^4_2 & 0 & b^2_2 b^1_1 & 0 & 0 \\ b^5_1 & b^5_2 & b^5_3 & -(b^4_1 b^1_2 + b^3_1 b^2_2 - b^4_2 b^1_1) & b^2_2 {b^1_1}^2 & b^2_2 b^1_2 b^1_1 \\ b^6_1 & b^6_2 & b^6_3 & -b^4_1 b^2_2 & 0 & {b^2_2}^2 b^1_1 \end{pmatrix}$$

where $b^2_2 b^1_1 \neq 0$. Taking suitable values for the $b^i_j$'s, we are led to the case where $\xi^2_1 = 1, \xi^3_1 = \xi^3_2 = \xi^5_5 = \xi^6_1 = \xi^6_3 = 0$ and moreover $\xi^6_2 = \xi^6_4 = 0$. Hence any $J$ in (59) is equivalent to :

$$J(\xi^3_3, \xi^4_3) = \begin{pmatrix} -\xi^3_3/\xi^4_3 & -({\xi^4_3}^2 + {\xi^3_3}^2)/{\xi^4_3}^2 & 0 & 0 & 0 & 0 \\ 1 & \xi^3_3/\xi^4_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^3_3 & -({\xi^3_3}^2 + 1)/\xi^4_3 & 0 & 0 \\ 0 & 0 & \xi^4_3 & -\xi^3_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(\xi^4_3 + 1)/\xi^4_3 \\ 0 & 0 & 0 & 0 & \xi^4_3/(\xi^4_3 + 1) & 0 \end{pmatrix} \tag{61}$$

where $\xi^4_3 \neq 0, -1$. $J(\xi^3_3, \xi^4_3) \cong J(\eta^3_3, \eta^4_3)$ if and only if $\xi^3_3 = \eta^3_3$ and $\xi^4_3 = \eta^4_3$. Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = -\xi^4_3\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = (\xi^4_3 + 1)\tilde{x}_5 - \xi^3_3\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = \frac{\xi^3_3(-{\xi^4_3}^2 + 1)}{{\xi^4_3}^2}\,\tilde{x}_5 + \frac{\xi^4_3 + {\xi^3_3}^2}{\xi^4_3}\,\tilde{x}_6$.

From (59), $\mathfrak{X}_{6,8}$ is a submanifold of dimension 10 in $\mathbb{R}^{36}$. It is the disjoint union of the continuously many orbits of the $J(\xi^3_3, \xi^4_3)$ defined in (61) where $\xi^4_3 \neq 0, -1$.

## 9.1

$$X_1 = \frac{\partial}{\partial x^1} - y^1\,\frac{\partial}{\partial y^2} + y^2\,\frac{\partial}{\partial x^3} + \frac{(y^1)^2}{2}\,\frac{\partial}{\partial y^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - x^2\frac{\partial}{\partial x^3} - y^2\frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J(\xi^3_3, \xi^4_3)$ defined in (61) where $\xi^4_3 \neq 0, -1$. Let $H_{\mathbb{C}}(G)$ the space of complex valued holomorphic functions on $G$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}^-_j\ f = 0\ \forall j = 1, 3, 5\}$. One has

$$\begin{aligned} \tilde{X}^-_1 &= 2\,\frac{\partial}{\partial \overline{w^1}} - y^1\left(1 - \frac{i\xi^3_3}{\xi^4_3}\right)\frac{\partial}{\partial y^2} - \left(y^2\left(1 - \frac{i\xi^3_3}{\xi^4_3}\right) + ix^2\right)\frac{\partial}{\partial x^3} + \left(\frac{(y^1)^2}{2}\left(1 - \frac{i\xi^3_3}{\xi^4_3}\right) - iy^2\right)\frac{\partial}{\partial y^3} \quad , \\ \tilde{X}^-_3 &= 2\frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}^-_5 = 2\,\frac{\partial}{\partial \overline{w^3}} \quad , \end{aligned}$$

where $w^1 = x^1 + \frac{\xi^3_3}{\xi^4_3}\,y^1 + iy^1 \quad , \quad w^2 = x^2 - \frac{\xi^3_3}{\xi^4_3}\,y^2 + \frac{i}{\xi^4_3}\,y^2 \quad , \quad w^3 = x^3 + i\,\frac{\xi^4_3 + 1}{\xi^4_3}\,y^3$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$2\,\frac{\partial f}{\partial \overline{w^1}} - iA\frac{w^1 - \overline{w^1}}{2}\,\frac{\partial f}{\partial w^2} - \left(iw^2 + B\,\frac{(w^1 - \overline{w^1})^2}{8}\right)\frac{\partial f}{\partial w^3} \quad = \quad 0$$

where

$$A = \frac{1}{\xi^4_3}\left(\xi^3_3\left(1 - \frac{1}{\xi^4_3}\right) - i\left(1 + \frac{{\xi^3_3}^2}{\xi^4_3}\right)\right) \quad , \quad B = \frac{\xi^4_3 + 1}{{\xi^4_3}^2}\,\left(\xi^3_3 + i\xi^4_3\right).$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1 \quad , \quad \varphi^2 = w^2 + \frac{iA}{4}\left(w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2}\right) \quad , \\
\varphi^3 &= w^3 + \frac{i}{2}\,\overline{w^1}w^2 - \frac{B}{48}\,(w^1 - \overline{w^1})^3 - \frac{A}{16}\,w^1(\overline{w^1})^2 + \frac{A}{24}\,(\overline{w^1})^3
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned}
w^1_{ax} &= w^1_a + w^1_x \quad , \quad w^2_{ax} = w^2_a + w^2_x + b^1x^1\,\frac{\xi^3_3 - i}{\xi^4_3} \quad , \\
w^3_{ax} &= w^3_a + w^3_x - b^2x^1 + b^1\,\frac{(x^1)^2}{2} - a^2y^1 + i\,\frac{\xi^4_3 + 1}{\xi^4_3}\left(\frac{(b^1)^2}{2}x^1 - (b^2 - b^1x^1)y^1\right).
\end{aligned}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x + \chi^2(a,x) \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi^3(a,x) \quad ,$$

where

$$\chi^2(a,x) = \frac{\xi^3_3 - i}{4{\xi^4_3}^2}\,\varphi^1_x\left(2i\xi^4_3\overline{\varphi^1_a} + (\xi^3_3 - i\xi^4_3)\varphi^1_a\right) \quad ,$$

$$\begin{aligned}
\chi^3(a,x) = \frac{i}{2}\,\varphi^2_x\overline{\varphi^1_a} &+ \frac{1}{16{\xi^4_3}^2}\,(\varphi^1_x)^2(4i{\xi^4_3}^2 - (\xi^3_3 - i)(3\xi^4_3 + i\xi^3_3))\left(\overline{\varphi^1_a} - \varphi^1_a\right) \\
&+ \varphi^1_x\left(\frac{1}{16\xi^4_3}\,(\overline{\varphi^1_a})^2(-\xi^3_3(\xi^4_3 - i\xi^3_3) - i\xi^4_3 - 3\xi^3_3 + 2i)\right. \\
&\quad + \frac{1}{16\xi^4_3}\,(\varphi^1_a)^2(-\xi^3_3(\xi^4_3 - i\xi^3_3) - i\xi^4_3 + \xi^3_3 - 2i) \\
&\quad \left. + \frac{\xi^3_3(\xi^4_3 - i\xi^3_3)}{4\xi^4_3}\,\overline{\varphi^1_a}\varphi^1_a - \frac{i\xi^4_3}{2}\,\overline{\varphi^2_a} + \frac{i(\xi^4_3 + 1)}{2}\,\varphi^2_a\right).
\end{aligned}$$

# 10 Lie Algebra $M10$.

Commutation relations for $M10$ : $[x_1, x_2] = x_3$; $[x_1, x_3] = x_5$; $[x_1, x_4] = x_6$; $[x_2, x_3] = -x_6$; $[x_2, x_4] = x_5$.

## 10.1 Case $\xi^4_3 \neq \xi^2_1$.

$$J = \begin{pmatrix}
\boxed{\xi^1_1} & -\frac{{\xi^1_1}^2+1}{\xi^2_1} & 0 & 0 & 0 & 0 \\
\boxed{\xi^2_1} & -\xi^1_1 & 0 & 0 & 0 & 0 \\
\boxed{\xi^3_1} & \frac{({\xi^3_3}^2+1)\xi^4_1 - (\xi^3_3+\xi^1_1)\xi^4_3\xi^3_1}{\xi^4_3\xi^2_1} & \boxed{\xi^3_3} & -\frac{{\xi^3_3}^2+1}{\xi^4_3} & 0 & 0 \\
\boxed{\xi^4_1} & \frac{(\xi^3_3-\xi^1_1)\xi^4_1 - \xi^4_3\xi^3_1}{\xi^2_1} & \boxed{\xi^4_3} & -\xi^3_3 & 0 & 0 \\
* & * & \boxed{\xi^5_3} & * & r & -\frac{r^2+1}{b} \\
\boxed{\xi^6_1} & \boxed{\xi^6_2} & \boxed{\xi^6_3} & * & b & -r
\end{pmatrix} \tag{62}$$

where $\mathbf{J^5_1} = (((2\xi^4_3\xi^1_1 - \xi^3_3\xi^2_1)\xi^2_1 + ({\xi^1_1}^2 + 1)\xi^3_3 - ({\xi^1_1}^2 + 1 + {\xi^2_1}^2)\xi^1_1)\xi^6_1{\xi^4_3}^2 + ((\xi^4_3{\xi^1_1}^2 + \xi^4_3 - 2\xi^3_3\xi^2_1\xi^1_1 - 2\xi^2_1)\xi^4_3 + ({\xi^3_3}^2 + 1){\xi^2_1}^2)\xi^5_3\xi^4_1 + ({\xi^4_3}^2\xi^2_1 - \xi^4_3{\xi^2_1}^2 - \xi^4_3{\xi^1_1}^2 - \xi^4_3 + {\xi^3_3}^2\xi^2_1 + \xi^2_1)\xi^6_2\xi^4_3\xi^2_1 - ((((\xi^3_3 - \xi^1_1)\xi^4_1 - \xi^4_3\xi^3_1)({\xi^3_3}^2 + 1) - ({\xi^4_3}^2\xi^3_1 - \xi^4_3\xi^4_1\xi^3_3 - \xi^4_3\xi^4_1\xi^1_1 + 2\xi^4_1\xi^3_3\xi^2_1)\xi^4_3)\xi^2_1 + ({\xi^1_1}^2 + 1 + {\xi^2_1}^2)\xi^3_3{\xi^4_3}^2\xi^3_1)\xi^6_3)/(((\xi^4_3\xi^1_1 - \xi^3_3\xi^2_1)^2 + (\xi^4_3 - \xi^2_1)^2)\xi^4_3)$; $\mathbf{J^5_2} = (\xi^6_3{\xi^4_3}^2\xi^4_1\xi^2_1{\xi^1_1}^2 + \xi^6_3{\xi^4_3}^2\xi^4_1\xi^2_1 - \xi^6_3{\xi^4_3}^2\xi^3_3\xi^3_1{\xi^2_1}^2 + \xi^6_3{\xi^4_3}^2\xi^3_3\xi^3_1{\xi^1_1}^2 + \xi^6_3{\xi^4_3}^2\xi^3_3\xi^3_1 + \xi^6_3{\xi^4_3}^2\xi^3_1{\xi^2_1}^2\xi^1_1 + \xi^6_3{\xi^4_3}^2\xi^3_1{\xi^1_1}^3 + \xi^6_3{\xi^4_3}^2\xi^3_1\xi^1_1 + \xi^6_3\xi^4_3\xi^4_1{\xi^3_3}^2{\xi^2_1}^2 - \xi^6_3\xi^4_3\xi^4_1{\xi^3_3}^2{\xi^1_1}^2 - \xi^6_3\xi^4_3\xi^4_1{\xi^3_3}^2 - 2\xi^6_3\xi^4_3\xi^4_1\xi^3_3{\xi^2_1}^2\xi^1_1 - \xi^6_3\xi^4_3\xi^4_1{\xi^2_1}^2 - \xi^6_3\xi^4_3\xi^4_1{\xi^1_1}^2 -$

$\xi_3^6\xi_3^4\xi_1^4 - 2\xi_3^6\xi_3^4{\xi_3^3}^2\xi_1^3\xi_1^2\xi_1^1 - 2\xi_3^6\xi_3^4\xi_1^3\xi_1^2\xi_1^1 + 2\xi_3^6\xi_1^4{\xi_3^3}^3\xi_1^2\xi_1^1 - \xi_3^6\xi_1^4{\xi_3^3}^2\xi_1^2{\xi_1^1}^2 + \xi_3^6\xi_1^4{\xi_3^3}^2\xi_1^2 + 2\xi_3^6\xi_1^4\xi_3^3\xi_1^2\xi_1^1 - \xi_3^6\xi_1^4\xi_1^2{\xi_1^1}^2 + \xi_3^6\xi_1^4\xi_1^2 - \xi_2^6{\xi_3^4}^2\xi_3^3{\xi_1^2}^3 + \xi_2^6{\xi_3^4}^2\xi_3^3\xi_1^2{\xi_1^1}^2 + \xi_2^6{\xi_3^4}^2\xi_3^3\xi_1^2 + \xi_2^6{\xi_3^4}^2{\xi_1^2}^3\xi_1^1 + \xi_2^6{\xi_3^4}^2\xi_1^2{\xi_1^1}^3 + \xi_2^6{\xi_3^4}^2\xi_1^2\xi_1^1 - 2\xi_2^6\xi_3^4{\xi_3^3}^2{\xi_1^2}^2\xi_1^1 - 2\xi_2^6\xi_3^4{\xi_1^2}^2\xi_1^1 - \xi_1^6{\xi_3^4}^3\xi_1^2{\xi_1^1}^2 - \xi_1^6{\xi_3^4}^3\xi_1^2 + \xi_1^6{\xi_3^4}^2{\xi_1^2}^2{\xi_1^1}^2 + \xi_1^6{\xi_3^4}^2{\xi_1^2}^2 + \xi_1^6{\xi_3^4}^2{\xi_1^1}^4 + 2\xi_1^6{\xi_3^4}^2{\xi_1^1}^2 + \xi_1^6{\xi_3^4}^2 - \xi_1^6\xi_3^4{\xi_3^3}^2\xi_1^2{\xi_1^1}^2 - \xi_1^6\xi_3^4{\xi_3^3}^2\xi_1^2 - \xi_1^6\xi_3^4\xi_1^2{\xi_1^1}^2 - \xi_1^6\xi_3^4\xi_1^2 - \xi_3^5{\xi_3^4}^3\xi_1^3{\xi_1^1}^2 - \xi_3^5{\xi_3^4}^3\xi_1^3 + \xi_3^5{\xi_3^4}^2\xi_1^4\xi_3^3{\xi_1^1}^2 + \xi_3^5{\xi_3^4}^2\xi_1^4\xi_3^3 - \xi_3^5{\xi_3^4}^2\xi_1^4{\xi_1^1}^3 - \xi_3^5{\xi_3^4}^2\xi_1^4\xi_1^1 + 2\xi_3^5{\xi_3^4}^2\xi_3^3\xi_1^3\xi_1^2\xi_1^1 + 2\xi_3^5{\xi_3^4}^2\xi_1^3\xi_1^2 - 2\xi_3^5\xi_3^4\xi_1^4{\xi_3^3}^2\xi_1^2\xi_1^1 + 2\xi_3^5\xi_3^4\xi_1^4\xi_3^3\xi_1^2{\xi_1^1}^2 - 2\xi_3^5\xi_3^4\xi_1^4\xi_3^3\xi_1^2 + 2\xi_3^5\xi_3^4\xi_1^4\xi_1^2\xi_1^1 - \xi_3^5\xi_3^4{\xi_3^3}^2\xi_1^3{\xi_1^2}^2 - \xi_3^5\xi_3^4\xi_1^3{\xi_1^2}^2 + \xi_3^5\xi_1^4{\xi_3^3}^3{\xi_1^2}^2 - \xi_3^5\xi_1^4{\xi_3^3}^2{\xi_1^2}^2\xi_1^1 + \xi_3^5\xi_1^4\xi_3^3{\xi_1^2}^2 - \xi_3^5\xi_1^4{\xi_1^2}^2\xi_1^1)/((\xi_3^4\xi_1^1 - \xi_3^3\xi_1^2)^2 + (\xi_3^4 - \xi_1^2)^2)\xi_3^4\xi_1^2)$; $\mathbf{J_4^5} = (-(\xi_3^6{\xi_3^4}^2{\xi_1^2}^2 + 2\xi_3^6\xi_3^4\xi_3^3\xi_1^2\xi_1^1 - 2\xi_3^6\xi_3^4\xi_1^2 + \xi_3^6{\xi_3^3}^2{\xi_1^1}^2 + \xi_3^6{\xi_3^3}^2 + \xi_3^6{\xi_1^1}^2 + \xi_3^6 + \xi_3^5{\xi_3^4}^2\xi_3^3\xi_1^2 - \xi_3^5{\xi_3^4}^2\xi_1^2\xi_1^1 - 2\xi_3^5\xi_3^4\xi_3^3{\xi_1^1}^2 - 2\xi_3^5\xi_3^4\xi_3^3 + \xi_3^5{\xi_3^3}^3\xi_1^2 + \xi_3^5{\xi_3^3}^2\xi_1^2\xi_1^1 + \xi_3^5\xi_3^3\xi_1^2 + \xi_3^5\xi_1^2\xi_1^1))/((((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + \xi_1^2({\xi_3^3}^2 + 1))\xi_3^4)$; $\mathbf{r} = \mathbf{J_5^5} = -(((\xi_3^4\xi_1^1 - \xi_3^3\xi_1^2)\xi_1^2 + ({\xi_1^1}^2 + 1)\xi_3^3)\xi_3^4 - ({\xi_3^3}^2 + 1)\xi_1^2\xi_1^1)/(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$; $\mathbf{J_4^6} = (-(((\xi_3^4\xi_3^3 + \xi_3^4\xi_1^1 - 2\xi_3^3\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1)(\xi_3^3 - \xi_1^1))\xi_3^6\xi_1^2 - ((\xi_3^4{\xi_1^1}^2 + \xi_3^4 - 2\xi_3^3\xi_1^2\xi_1^1 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)\xi_3^5))/((({\xi_3^4}^2 - \xi_3^4\xi_1^2 + {\xi_3^3}^2 + 1)\xi_1^2 - ({\xi_1^1}^2 + 1)\xi_3^4)\xi_3^4)$; $\mathbf{b} = \mathbf{J_5^6} = -((\xi_3^4{\xi_1^1}^2 + \xi_3^4 - 2\xi_3^3\xi_1^2\xi_1^1 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)/(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$; and the parameters are subject to the condition

$$\begin{aligned} \xi_1^2\xi_3^4(\xi_1^2 - \xi_3^4) &\neq 0 && (63)\\ ((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2 &\neq 0. && (64) \end{aligned}$$

Note that $b \neq 0$ since $\mathrm{Num}(b) = -\|Y - Z\|^2$ where $Y = \xi_3^4 \begin{pmatrix} \xi_1^1 \\ 1 \end{pmatrix}$ , $Z = \xi_1^2 \begin{pmatrix} \xi_3^3 \\ 1 \end{pmatrix}$.

Now the automorphism group of $M10$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_1^2 u & 0 & 0 & 0 & 0 \\ b_1^2 & -b_1^1 u & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & -({b_1^2}^2 + {b_1^1}^2)u & 0 & 0 & 0 \\ b_1^4 & b_2^4 & 0 & {b_1^2}^2 + {b_1^1}^2 & 0 & 0 \\ b_1^5 & b_2^5 & b_2^3 b_1^1 - b_1^3 b_1^2 u + b_1^4 b_1^1 u + b_2^4 b_1^2 & b_4^5 & -({b_1^2}^2 + {b_1^1}^2)b_1^1 u & ({b_1^2}^2 + {b_1^1}^2)b_1^2 \\ b_1^6 & b_2^6 & -(b_2^3 b_1^2 + b_1^3 b_1^1 u + b_1^4 b_1^2 u - b_2^4 b_1^1) & b_4^6 & ({b_1^2}^2 + {b_1^1}^2)b_1^2 u & ({b_1^2}^2 + {b_1^1}^2)b_1^1 \end{pmatrix} \quad (65)$$

where $u^2 = 1$ and ${b_1^2}^2 + {b_1^1}^2 \neq 0$. Equivalence by a suitable block-diagonal automorphism (65) leads to the case $\xi_1^1 = 0$ and we then can suppose $0 < \xi_1^2 \leqslant 1$. Equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \xi_1^3/\xi_1^2 & 1 & 0 & 0 & 0 \\ 0 & \xi_1^4/\xi_1^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & \xi_1^3/\xi_1^2 & 0 & 1 & 0 \\ 0 & 0 & \xi_1^4/\xi_1^2 & 0 & 0 & 1 \end{pmatrix} \quad (66)$$

leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$. Then, equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ b_1^5 & b_2^5 & 0 & \xi_3^5/\xi_3^4 & 1 & 0 \\ 0 & 0 & 0 & \xi_3^6/\xi_3^4 & 0 & 1 \end{pmatrix}$$

with suitable $b_1^5, b_2^5$ leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0, \xi_3^5 = 0, \xi_1^6 = 0, \xi_2^6 = 0, \xi_3^6 = 0:$

$$J(\xi_1^2, \xi_3^3, \xi_3^4) = \mathrm{diag}\left( \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} \xi_3^3 & -\frac{{\xi_3^3}^2 + 1}{\xi_3^4} \\ \xi_3^4 & -\xi_3^3 \end{pmatrix}, \begin{pmatrix} r & -\frac{1 + r^2}{b} \\ b & -r \end{pmatrix} \right) \quad (67)$$

where

$$0 < \xi_1^2 \leqslant 1; \quad \xi_1^2\xi_3^4(\xi_1^2 - \xi_3^4) \neq 0; \quad (\xi_3^4\xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2 \neq 0 \quad (68)$$

and $r = ((\xi_1^2 + 1)(\xi_1^2 - 1)\xi_3^4\xi_3^3)/((\xi_3^4\xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$,
$b = -((\xi_3^4 - 2\xi_1^2)\xi_3^4 + ({\xi_3^3}^2 + 1){\xi_1^2}^2)/((\xi_3^4\xi_1^2 - {\xi_1^2}^2 - 1)\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2)$.
Now suppose $J(\eta_1^2, \eta_3^3, \eta_3^4) = \Phi^{-1} J(\xi_1^2, \xi_3^3, \xi_3^4)\Phi$ where $\Phi$ is given in (65) and the $\eta$'s and $\xi$'s satisfy (68). Computing the matrix $J2 = \Phi^{-1} J(\xi_1^2, \xi_3^3, \xi_3^4)\Phi$, one gets $J2_1^1 = (b_1^2 b_1^1({\xi_1^2}^2 - 1))/(\xi_1^2({b_1^2}^2 + {b_1^1}^2))$, $J2_1^2 = -({b_1^2}^2 + {b_1^1}^2{\xi_1^2}^2)/(\xi_1^2 u({b_1^2}^2 + {b_1^1}^2))$, $J2_3^3 = \xi_3^3$, $J2_3^4 = -\xi_3^4 u$. From these formulae, we see that a necessary condition for equivalence is that $\eta_3^4 = -u\xi_3^4$ $(u = \pm 1)$ and $\eta_3^3 = \xi_3^3$. As $u = 1$ would change the sign of $\xi_1^2$, we conclude that $u = -1$. Now to keep $J2_1^1 = 0$, one must have either $\xi_1^2 = 1$

or $b_1^1 b_1^2 = 0$. If $\xi_1^2 = 1$, or if $\xi_1^2 < 1$ and $b_1^2 = 0$, then $\eta_1^2 = \xi_1^2$ and $\eta_3^3 = \xi_3^3, \eta_3^4 = \xi_3^4$. If $\xi_1^2 < 1$ and $b_1^1 = 0$, then $\eta_1^2 = 1/\xi_1^2 > 1$ which is contradictory. Hence $J(\eta_1^2, \eta_3^3, \eta_3^4)$ and $J(\xi_1^2, \xi_3^3, \xi_3^4)$ are not equivalent unless $\eta_1^2 = \xi_1^2, \eta_3^3 = \xi_3^3, \eta_3^4 = \xi_3^4$.

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = (-\xi_3^4 \xi_1^2 + 1)\tilde{x}_5 + \xi_3^3 \xi_1^2 \tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_4] = \xi_3^3 \xi_1^2 \tilde{x}_5 + \frac{\xi_3^4 - {\xi_3^3}^2 \xi_1^2 - \xi_1^2}{\xi_3^4}\,\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \frac{\xi_3^3}{\xi_1^2}\,\tilde{x}_5 + \frac{\xi_3^4 - \xi_1^2}{\xi_1^2}\,\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_4] = \frac{\xi_3^4 \xi_1^2 - {\xi_3^3}^2 - 1}{\xi_3^4 \xi_1^2}\,\tilde{x}_5 - \frac{\xi_3^3}{\xi_1^2}\,\tilde{x}_6$.

## 10.2 Case $\xi_3^4 = \xi_1^2, \xi_3^3 = \xi_1^1$.

In that case one has necessarily $\xi_1^1 = 0$.

$$J = \begin{pmatrix} 0 & -1/\xi_1^2 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & 0 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \xi_1^4 & 0 & -1/\xi_1^2 & 0 & 0 \\ \boxed{\xi_1^4} & -\xi_1^3 & \xi_1^2 & 0 & 0 & 0 \\ * & * & \boxed{\xi_3^5} & (({\xi_5^5}^2+1)\xi_3^6 - \xi_5^6 \xi_5^5 \xi_3^5)/(\xi_5^6 \xi_1^2) & \boxed{\xi_5^5} & -({\xi_5^5}^2+1)/\xi_5^6 \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & -(\xi_5^6 \xi_3^5 - \xi_3^6 \xi_5^5)/\xi_1^2 & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{69}$$

where $\mathbf{J_1^5} = (-(\xi_2^6 - \xi_1^6 \xi_5^5 \xi_1^2 + (\xi_5^5 \xi_1^4 + \xi_1^3 \xi_1^2)\xi_3^6 - \xi_5^6 \xi_3^5 \xi_1^4))/(\xi_5^6 \xi_1^2)$; $\mathbf{J_2^5} = (\xi_2^6 \xi_5^5 + \xi_1^6 \xi_1^2 + (\xi_5^5 \xi_1^3 \xi_1^2 - \xi_1^4)\xi_3^6 - \xi_5^6 \xi_3^5 \xi_1^3 \xi_1^2)/\xi_5^6$; and the parameters are subject to the condition

$$\xi_1^2 = \pm 1, \quad \xi_5^6 \neq 0. \tag{70}$$

As in the preceding case, by equivalence by a suitable block-diagonal automorphism, we can suppose $\xi_5^5 = 0,\ 0 < \xi_5^6 \leqslant 1$ and then equivalence by (66) leads to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ (\xi_1^2 - \xi_1^6)/\xi_5^6 & \xi_2^6 \xi_1^2/(\xi_5^6 \xi_1^2) & 0 & \xi_3^5/\xi_1^2 & 1 & 0 \\ 0 & 0 & 0 & \xi_3^6/\xi_1^2 & 0 & 1 \end{pmatrix}$$

leads to the case where moreover $\xi_3^5 = 0, \xi_1^6 = 0, \xi_2^6 = 0, \xi_3^6 = 0$ :

$$J(\xi_1^2, \xi_5^6) = \operatorname{diag}\left(\begin{pmatrix} 0 & -\frac{1}{\xi_1^2} \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\xi_5^6} \\ \xi_5^6 & 0 \end{pmatrix}\right) \tag{71}$$

and the parameters are subject to the condition

$$\xi_1^2 = \pm 1,\ 0 < \xi_5^6 \leqslant 1. \tag{72}$$

As in the preceding case, we can see that $J(\eta_1^2, \eta_5^6)$ and $J(\xi_1^2, \xi_5^4)$ are not equivalent unless $\eta_1^2 = \xi_1^2, \eta_5^6 = \xi_5^6$. Commutation relations of $\mathfrak{m}$ : $\mathfrak{m}$ is abelian. Since $\mathfrak{m}$ is abelian, no $J(\xi_1^2, \xi_5^6)$ is equivalent to any $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (67).

## 10.3 Case $\xi_3^4 = \xi_1^2, \xi_3^3 \neq \xi_1^1$.

In that case one has necessarily $\xi_3^3 \neq -\xi_1^1$ as well.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & -\frac{{\xi_1^1}^2+1}{k} & 0 & 0 & 0 & 0 \\ k & -\xi_1^1 & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & \frac{(({\xi_3^3}^2+1)(\xi_3^3-\xi_1^1)\xi_1^4+(\xi_3^3+\xi_1^1)^2\xi_5^6\xi_1^3)(\xi_3^3-\xi_1^1)}{(\xi_3^3+\xi_1^1)^2{\xi_5^6}^2} & \boxed{\xi_3^3} & -\frac{{\xi_3^3}^2+1}{k} & 0 & 0 \\ \boxed{\xi_1^4} & -\frac{(\xi_3^3+\xi_1^1)\xi_5^6\xi_1^3+(\xi_3^3-\xi_1^1)^2\xi_1^4}{(\xi_3^3+\xi_1^1)\xi_5^6} & k & -\xi_3^3 & 0 & 0 \\ * & * & \boxed{\xi_3^5} & * & m & -\frac{m^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & * & \boxed{\xi_5^6} & -m \end{pmatrix} \tag{73}$$

where $\mathbf{k} = \mathbf{J^2_1} = -((\xi_3^3 + \xi_1^1)\xi_5^6)/(\xi_3^3 - \xi_1^1)$; $\mathbf{J^5_1} = ((({\xi_5^6}^3\xi_1^6{\xi_3^3}^2 + 2{\xi_5^6}^3\xi_1^6\xi_3^3\xi_1^1 + {\xi_5^6}^3\xi_1^6{\xi_1^1}^2 + {\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^2 - {\xi_5^6}^2\xi_3^6\xi_1^4{\xi_1^1}^2 + {\xi_5^6}^2\xi_2^6{\xi_3^3}^3 + {\xi_5^6}^2\xi_2^6{\xi_3^3}^2\xi_1^1 - {\xi_5^6}^2\xi_2^6\xi_3^3{\xi_1^1}^2 - {\xi_5^6}^2\xi_2^6{\xi_1^1}^3 - \xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3 + \xi_5^6\xi_3^6{\xi_3^3}^2\xi_1^3\xi_1^1 + \xi_5^6\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^2 - \xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^3 - \xi_5^6\xi_1^6{\xi_3^3}^3\xi_1^1 + \xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^2 + \xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^3 - \xi_5^6\xi_1^6{\xi_1^1}^4 - \xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^3 + 3\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^2\xi_1^1 - 3\xi_5^6\xi_3^5\xi_1^4\xi_3^3{\xi_1^1}^2 + \xi_5^6\xi_3^5\xi_1^4{\xi_1^1}^3 - \xi_3^6\xi_1^4{\xi_3^3}^4 + 3\xi_3^6\xi_1^4{\xi_3^3}^3\xi_1^1 - 3\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^2 + \xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^3)(\xi_3^3 + \xi_1^1) + (\xi_5^6\xi_1^6\xi_3^3 + \xi_5^6\xi_1^6\xi_1^1 + \xi_3^6\xi_1^4\xi_3^3 - \xi_3^6\xi_1^4\xi_1^1)(\xi_3^3\xi_1^1 - 1)(\xi_3^3 - \xi_1^1)^2)/((\xi_3^3 + \xi_1^1)^2(\xi_3^3 - \xi_1^1)^2{\xi_5^6}^2)$; $\mathbf{J^5_2} = ({\xi_5^6}^4\xi_2^6{\xi_3^3}^4 + 4{\xi_5^6}^4\xi_2^6{\xi_3^3}^3\xi_1^1 + 6{\xi_5^6}^4\xi_2^6{\xi_3^3}^2{\xi_1^1}^2 + 4{\xi_5^6}^4\xi_2^6\xi_3^3{\xi_1^1}^3 + {\xi_5^6}^4\xi_2^6{\xi_1^1}^4 - {\xi_5^6}^3\xi_3^6{\xi_3^3}^4\xi_1^3 - 2{\xi_5^6}^3\xi_3^6{\xi_3^3}^3\xi_1^3\xi_1^1 + 2{\xi_5^6}^3\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^3 + {\xi_5^6}^3\xi_3^6\xi_1^3{\xi_1^1}^4 - {\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^5 + {\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^4\xi_1^1 + 2{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^2 - 2{\xi_5^6}^2\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^3 - {\xi_5^6}^2\xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^4 + {\xi_5^6}^2\xi_3^6\xi_1^4{\xi_1^1}^5 + 2{\xi_5^6}^2\xi_2^6{\xi_3^3}^5\xi_1^1 + {\xi_5^6}^2\xi_2^6{\xi_3^3}^4{\xi_1^1}^2 - {\xi_5^6}^2\xi_2^6{\xi_3^3}^4 - 4{\xi_5^6}^2\xi_2^6{\xi_3^3}^3{\xi_1^1}^3 - 2{\xi_5^6}^2\xi_2^6{\xi_3^3}^2{\xi_1^1}^4 + 2{\xi_5^6}^2\xi_2^6{\xi_3^3}^2{\xi_1^1}^2 + 2{\xi_5^6}^2\xi_2^6\xi_3^3{\xi_1^1}^5 + {\xi_5^6}^2\xi_2^6{\xi_1^1}^6 - {\xi_5^6}^2\xi_2^6{\xi_1^1}^4 + {\xi_5^6}^2\xi_3^5{\xi_3^3}^5\xi_1^3 - {\xi_5^6}^2\xi_3^5{\xi_3^3}^4\xi_1^3\xi_1^1 - 2{\xi_5^6}^2\xi_3^5{\xi_3^3}^3\xi_1^3{\xi_1^1}^2 + 2{\xi_5^6}^2\xi_3^5{\xi_3^3}^2\xi_1^3{\xi_1^1}^3 + {\xi_5^6}^2\xi_3^5\xi_3^3\xi_1^3{\xi_1^1}^4 - {\xi_5^6}^2\xi_3^5\xi_1^3{\xi_1^1}^5 - 2\xi_5^6\xi_3^6{\xi_3^3}^5\xi_1^3\xi_1^1 + 3\xi_5^6\xi_3^6{\xi_3^3}^4\xi_1^3{\xi_1^1}^2 + \xi_5^6\xi_3^6{\xi_3^3}^4\xi_1^3 + 2\xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3{\xi_1^1}^3 - 2\xi_5^6\xi_3^6{\xi_3^3}^3\xi_1^3\xi_1^1 - 4\xi_5^6\xi_3^6{\xi_3^3}^2\xi_1^3{\xi_1^1}^4 + 2\xi_5^6\xi_3^6\xi_3^3\xi_1^3{\xi_1^1}^3 + \xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^6 - \xi_5^6\xi_3^6\xi_1^3{\xi_1^1}^4 - \xi_5^6\xi_1^6{\xi_3^3}^5{\xi_1^1}^2 - \xi_5^6\xi_1^6{\xi_3^3}^5 + \xi_5^6\xi_1^6{\xi_3^3}^4{\xi_1^1}^3 + \xi_5^6\xi_1^6{\xi_3^3}^4\xi_1^1 + 2\xi_5^6\xi_1^6{\xi_3^3}^3{\xi_1^1}^4 + 2\xi_5^6\xi_1^6{\xi_3^3}^3{\xi_1^1}^2 - 2\xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^5 - 2\xi_5^6\xi_1^6{\xi_3^3}^2{\xi_1^1}^3 - \xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^6 - \xi_5^6\xi_1^6\xi_3^3{\xi_1^1}^4 + \xi_5^6\xi_1^6{\xi_1^1}^7 + \xi_5^6\xi_1^6{\xi_1^1}^5 + \xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^6 - 4\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^5\xi_1^1 + 5\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^4{\xi_1^1}^2 - 5\xi_5^6\xi_3^5\xi_1^4{\xi_3^3}^2{\xi_1^1}^4 + 4\xi_5^6\xi_3^5\xi_1^4\xi_3^3{\xi_1^1}^5 - \xi_5^6\xi_3^5\xi_1^4{\xi_1^1}^6 - 2\xi_3^6\xi_1^4{\xi_3^3}^6\xi_1^1 + 8\xi_3^6\xi_1^4{\xi_3^3}^5{\xi_1^1}^2 - 12\xi_3^6\xi_1^4{\xi_3^3}^4{\xi_1^1}^3 - 2\xi_3^6\xi_1^4{\xi_3^3}^4\xi_1^1 + 8\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^4 + 8\xi_3^6\xi_1^4{\xi_3^3}^3{\xi_1^1}^2 - 2\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^5 - 12\xi_3^6\xi_1^4{\xi_3^3}^2{\xi_1^1}^3 + 8\xi_3^6\xi_1^4\xi_3^3{\xi_1^1}^4 - 2\xi_3^6\xi_1^4{\xi_1^1}^5)/((\xi_3^3 + \xi_1^1)^3(\xi_3^3 - \xi_1^1)^2{\xi_5^6}^3)$; $\mathbf{J^5_4} = ((({\xi_5^6}^3\xi_3^5\xi_3^3 + {\xi_5^6}^3\xi_3^5\xi_1^1 + \xi_5^6\xi_3^5{\xi_3^3}^3 - 2\xi_5^6\xi_3^5{\xi_3^3}^2\xi_1^1 + \xi_5^6\xi_3^5\xi_3^3{\xi_1^1}^2 - \xi_3^6{\xi_3^3}^2 + 2\xi_3^6\xi_3^3\xi_1^1 - \xi_3^6{\xi_1^1}^2)(\xi_3^3 + \xi_1^1) + (\xi_3^3\xi_1^1 - 1)(\xi_3^3 - \xi_1^1)^2\xi_5^6\xi_3^5)(\xi_3^3 + \xi_1^1)(\xi_3^3 - \xi_1^1)^2 - ((\xi_3^3\xi_1^1 - 1)(\xi_3^3 - \xi_1^1)^2 + (\xi_3^3 + \xi_1^1)^2{\xi_5^6}^2)^2\xi_3^6)/((\xi_3^3 + \xi_1^1)^3(\xi_3^3 - \xi_1^1)^3{\xi_5^6}^2)$; $\mathbf{m} = \mathbf{J^5_5} = ((\xi_3^3\xi_1^1 - 1)(\xi_3^3 - \xi_1^1)^2 + (\xi_3^3 + \xi_1^1)^2{\xi_5^6}^2)/((\xi_3^3 + \xi_1^1)(\xi_3^3 - \xi_1^1)^2)$; $\mathbf{J^6_4} = (-(({\xi_5^6}^2\xi_3^6\xi_3^3 + {\xi_5^6}^2\xi_3^6\xi_1^1 - \xi_5^6\xi_3^5{\xi_3^3}^2 + 2\xi_5^6\xi_3^5\xi_3^3\xi_1^1 - \xi_5^6\xi_3^5{\xi_1^1}^2 - \xi_3^6{\xi_3^3}^3 + 2\xi_3^6{\xi_3^3}^2\xi_1^1 - \xi_3^6\xi_3^3{\xi_1^1}^2)(\xi_3^3 + \xi_1^1) + (\xi_3^3\xi_1^1 - 1)(\xi_3^3 - \xi_1^1)^2\xi_3^6))/((\xi_3^3 + \xi_1^1)^2(\xi_3^3 - \xi_1^1)\xi_5^6)$; and the parameters are subject to the condition

$$\xi_3^3 \neq \pm\xi_1^1,\ \xi_1^2\xi_4^3\xi_5^6 \neq 0. \tag{74}$$

As in the preceding cases, equivalence by a suitable block-diagonal automorphism leads to the case $\xi_1^1 = 0$. Equivalences by successively

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\xi_1^3/\xi_5^6 & 1 & 0 & 0 & 0 \\ 0 & -\xi_1^4/\xi_5^6 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\xi_1^3/\xi_5^6 & 0 & 1 & 0 \\ 0 & 0 & -\xi_1^4/\xi_5^6 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\xi_1^6\xi_3^3/(\xi_5^6\xi_3^3) & -\xi_5^6\xi_2^6\xi_3^3/({\xi_5^6}^2\xi_3^3) & 0 & -\xi_3^5/\xi_5^6 & 1 & 0 \\ 0 & 0 & 0 & -\xi_3^6/\xi_5^6 & 0 & 1 \end{pmatrix}$$

lead to the case where moreover $\xi_1^3 = 0, \xi_1^4 = 0$ and $\xi_3^5 = 0, \xi_1^6 = 0, \xi_2^6 = 0, \xi_3^6 = 0$ :

$$J = \begin{pmatrix} 0 & 1/\xi_5^6 & 0 & 0 & 0 & 0 \\ -\xi_5^6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3^3 & ({\xi_3^3}^2 + 1)/\xi_5^6 & 0 & 0 \\ 0 & 0 & -\xi_5^6 & -\xi_3^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\xi_5^6 + 1)(\xi_5^6 - 1)/\xi_3^3 & -(({\xi_5^6}^2 - 2){\xi_5^6}^2 + {\xi_3^3}^2 + 1)/(\xi_5^6{\xi_3^3}^2) \\ 0 & 0 & 0 & 0 & \xi_5^6 & -(\xi_5^6 + 1)(\xi_5^6 - 1)/\xi_3^3 \end{pmatrix} \tag{75}$$

Computing the matrix $J2 = \Phi^{-1}J\Phi$ where $\Phi$ is given in (65) and $b_1^2 = b_1^1 = 1$, one gets $J2_1^2 = \frac{1+{\xi_5^6}^2}{2\xi_5^6}u$, $J2_3^4 = \xi_5^6 u$. Hence if ${\xi_5^6}^2 \neq 1$ we are back to case 1 (10.1). Finally, if ${\xi_5^6}^2 = 1$, equivalence by $\Phi = \operatorname{diag}(1, -1, -1, 1, -1, 1)$ leads to the case where moreover $\xi_5^6 = 1$ :

$$J(\xi_3^3) = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \xi_3^3 & {\xi_3^3}^2 + 1 \\ -1 & -\xi_3^3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \qquad (\xi_3^3 \neq 0). \tag{76}$$

$J(\xi_3^3)$ and $J(\eta_3^3)$ are non equivalent unless $\eta_3^3 = \xi_3^3$, and $J(\xi_3^3)$ in (76) is equivalent neither to any $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (67) nor to any $J(\xi_1^2, \xi_5^6)$ in (71).

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = -\xi_3^3\tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_4] = -\xi_3^3\tilde{x}_5 - {\xi_3^3}^2\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = -\xi_3^3\tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_4] = -{\xi_3^3}^2\tilde{x}_5 + \xi_3^3\tilde{x}_6$.

## 10.4 Conclusions.

To solve the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ in the 3 cases 1 (10.1), 2.1 (10.2), 2.2 (10.3), one has to complete first a set of common steps, and then we are left with solving the system $S$ of the remaining equations in the 12 variables

$\xi_1^1, \xi_1^2, \xi_1^3, \xi_3^3, \xi_1^4, \xi_3^4, \xi_3^5, \xi_5^5, \xi_1^6, \xi_2^6, \xi_3^6, \xi_5^6$ in the open subset $\xi_5^6\xi_3^4\xi_1^2 \neq 0$ of $\mathbb{R}^{12}$. Among these equations, we single out the 2 equations 13|6 and 14|6 which read:

$$\begin{cases} f = f_{13|6} = 0 \\ g = f_{14|6} = 0 \end{cases} \tag{77}$$

where : $f_{13|6} = \xi_5^6(\xi_3^3 + \xi_1^1) + \xi_5^5(\xi_1^2 - \xi_3^4) - \xi_3^4\xi_1^1 + \xi_3^3\xi_1^2$ and $f_{14|6} = (\xi_5^6(\xi_3^4\xi_1^2 - {\xi_3^3}^2 - 1) + \xi_5^5\xi_3^4(\xi_3^3 - \xi_1^1) + \xi_3^4\xi_3^3\xi_1^1 + \xi_3^4 - ({\xi_3^3}^2 + 1)\xi_1^2)/\xi_3^4$. In each of the 3 cases, the remaining system is equivalent to the system (77). To conclude that $\mathfrak{X}_{M10}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$, it will be sufficient to prove that the preceding system is of maximal rank 2 at any point of $\mathfrak{X}_{M10}$, that is in each of the 3 cases some 2-jacobian doesn't vanish. In case 1, one has $\frac{D(f,g)}{D(\xi_5^5,\xi_5^6)} = -\frac{1}{\xi_3^4}(((\xi_3^4 - \xi_1^2)\xi_1^2 - ({\xi_1^1}^2 + 1))\xi_3^4 + ({\xi_3^3}^2 + 1)\xi_1^2) \neq 0$. In case 2.1, one has $\frac{D(f,g)}{D(\xi_1^1,\xi_1^2)} = (\xi_5^6 - \xi_1^2)^2 + {\xi_5^5}^2 \neq 0$ if $\xi_5^5 \neq 0$. If $\xi_5^5 = 0$, $\frac{D(f,g)}{D(\xi_1^1,\xi_1^2)} = (\xi_5^6 - \xi_1^2)^2$ and $\frac{D(f,g)}{D(\xi_3^3,\xi_3^4)} = (\xi_5^6 + \xi_1^2)^2$. These 2-jacobians cannot simultaneously vanish. In case 2.2, one has $\frac{D(f,g)}{D(\xi_1^2,\xi_5^5)} = (\xi_3^3 - \xi_1^1)(\xi_5^5 + \xi_3^3) \neq 0$ if $\xi_5^5 \neq -\xi_3^3$. If $\xi_5^5 = -\xi_3^3$, $\frac{D(f,g)}{D(\xi_5^5,\xi_5^6)} = {\xi_1^1}^2 - {\xi_3^3}^2 \neq 0$. Hence the system (77) is of maximal rank 2 at any point of $\mathfrak{X}_{M10}$, and $\mathfrak{X}_{M10}$ is a 10-dimensional submanifold of $\mathbb{R}^{36}$.

Any CS is equivalent to one and only one of the following : $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (67) or $J(\xi_1^2, \xi_5^6)$ in (71) or $J(\xi_3^3)$ in (76).

## 10.5

$$X_1 = \frac{\partial}{\partial x^1} - y^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^3} - \left(y^2 + \frac{(y^1)^2}{2}\right)\frac{\partial}{\partial y^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - y^2\frac{\partial}{\partial x^3} + x^2\frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J(\xi_1^2, \xi_3^3, \xi_3^4)$ in (67) with conditions (68). Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1, 3, 5\}$. One has

$$\begin{aligned}
\tilde{X}_1^- &= 2\frac{\partial}{\partial \overline{w^1}} - y^1\frac{\partial}{\partial x^2} - \left(x^2 + i\xi_1^2 y^2\right)\frac{\partial}{\partial x^3} - \left(\frac{(y^1)^2}{2} + y^2 - i\xi_1^2 x^2\right)\frac{\partial}{\partial y^3} \quad , \\
\tilde{X}_3^- &= 2\frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}_5^- = 2\frac{\partial}{\partial \overline{w^3}} \quad ,
\end{aligned}$$

where

$$w^1 = x^1 + \frac{i}{\xi_1^2}y^1 \quad , \quad w^2 = x^2 - \frac{\xi_3^3}{\xi_3^4}y^2 + \frac{i}{\xi_3^4}y^2 \quad , \quad w^3 = x^3 - \frac{r}{b}y^3 + \frac{i}{b}y^3.$$

Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$\begin{aligned}
2\frac{\partial f}{\partial \overline{w^1}} + i\xi_1^2\frac{w^1 - \overline{w^1}}{2i}\frac{\partial f}{\partial w^2} + \frac{1}{8b}\Big[{\xi_1^2}^2(r - i)\left(2\overline{w^1}w^1 - (\overline{w^1})^2 - {w^1}^2\right) \\
-4\overline{w^2}\left(i(c + \xi_1^2 r) + \xi_1^2 + b\right) - 4w^2\left(i(-c + \xi_1^2 r) + \xi_1^2 + b\right)\Big]\frac{\partial f}{\partial w^3} = 0.
\end{aligned}$$

where $c = i\xi_3^4\xi_1^2 b - \xi_3^4 r + i\xi_3^4 + i\xi_3^3\xi_1^2 r + \xi_3^3\xi_1^2 + b\xi_3^3$. The 3 functions

$$\varphi^1 = w^1 \quad , \quad \varphi^2 = w^2 - \frac{i\xi_1^2}{4}\left(w^1\overline{w^1} - \frac{(\overline{w^1})^2}{2}\right) \quad ,$$

$$\begin{aligned}
\varphi^3 = w^3 + \frac{1}{48b}\Big((\overline{w^1})^3\xi_1^2(c + ib) + 3(\overline{w^1})^2 w^1{\xi_1^2}^2(i - r) + 12\overline{w^1}\overline{w^2}(ic + i\xi_1^2 r + \xi_1^2 + b) \\
+3\overline{w^1}{w^1}^2\xi_1^2(-c + 2\xi_1^2 r - 2i\xi_1^2 - ib) + 12(\overline{w^1} + w^1)w^2(-ic + i\xi_1^2 r + \xi_1^2 + b)\Big)
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned} w^1_{ax} &= w^1_a + w^1_x \\ w^2_{ax} &= w^2_a + w^2_x - b^1 x^1 \\ w^3_{ax} &= w^3_a + w^3_x - a^2 x^1 + \frac{1}{2} b^1 (x^1)^2 - b^2 y^1 + \frac{i-r}{b} \left( -b^2 x^1 - \frac{1}{2} (b^1)^2 x^1 + y^1 (a^2 - b^1 x^1) \right). \end{aligned}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x - \frac{i\xi^2_1}{4} (2\overline{\varphi^1_a} - \varphi^1_a)\, \varphi^1_x \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi^3(a,x)$$

where

$$\begin{aligned} \chi^3(a,x) = {} & \frac{\xi^2_1}{16((\xi^4_3 - \xi^2_1)^2 + {\xi^3_3}^2 {\xi^2_1}^2)} D_1\, (\varphi^1_x)^2 \\ & + \frac{1}{16(\xi^4_3 - i\xi^3_3\xi^2_1 - \xi^2_1)} \left( D_2 \xi^2_1\, \varphi^1_x + 8\xi^4_3 (1 - {\xi^2_1}^2) \left( \overline{\varphi^1_a} + \varphi^1_a \right) \varphi^2_x \right) \end{aligned}$$

with
$D_1 = i\overline{\varphi^1_a}\, (7{\xi^4_3}^2 {\xi^2_1}^2 - 3{\xi^4_3}^2 + 7i\xi^4_3\xi^3_3{\xi^2_1}^3 - 7i\xi^4_3\xi^3_3\xi^2_1 - 7\xi^4_3{\xi^2_1}^3 - \xi^4_3\xi^2_1 + 4{\xi^3_3}^2{\xi^2_1}^2 + 4{\xi^2_1}^2) + i\varphi^1_a\, (-4{\xi^4_3}^2{\xi^2_1}^2 + {\xi^4_3}^2 - 4i\xi^4_3\xi^3_3{\xi^2_1}^3 + 4i\xi^4_3\xi^3_3\xi^2_1 + 4\xi^4_3{\xi^2_1}^3 + 2\xi^4_3\xi^2_1 - 3{\xi^3_3}^2{\xi^2_1}^2 - 3{\xi^2_1}^2)$;
$D_2 = -i(\overline{\varphi^1_a})^2\, ({\xi^4_3}^2\xi^2_1 - 2\xi^4_3{\xi^2_1}^2 + 2\xi^4_3 + {\xi^3_3}^2\xi^2_1 - 2i\xi^3_3\xi^2_1 - \xi^2_1) + 4i\overline{\varphi^1_a}\varphi^1_a\, ({\xi^4_3}^2\xi^2_1 + \xi^4_3{\xi^2_1}^2 - \xi^4_3 + {\xi^3_3}^2\xi^2_1 - 2i\xi^3_3\xi^2_1 - \xi^2_1) - 8\overline{\varphi^2_a}\, ({\xi^4_3}^2 + {\xi^3_3}^2 - 2i\xi^3_3 - 1) + 8\varphi^2_a\, ({\xi^4_3}^2 + {\xi^3_3}^2 - 2\xi^3_3\xi^2_1 + 1) - i{\varphi^1_a}^2\, ({\xi^4_3}^2\xi^2_1 + 3\xi^4_3{\xi^2_1}^2 - 2\xi^4_3 + {\xi^3_3}^2\xi^2_1 - 3i\xi^3_3\xi^2_1 - 2\xi^2_1)$.

In the case of $J(\xi^2_1, \xi^6_5)$ in (71) where $\xi^2_1 = \pm 1$, and $0 < \xi^6_5 \leqslant 1$, the preceding computations apply with $r = 0, b = \xi^6_5, \xi^3_3 = 0, \xi^4_3 = \xi^2_1$. The only difference is that we get now

$$\chi^3(a,x) = \frac{1}{16\xi^6_5} \left( D_1\, (\varphi^1_x)^2 + D_2\, \varphi^1_x \right) + \frac{\xi^6_5 + \xi^2_1}{2\xi^6_5} \left( \overline{\varphi^1_a} + \varphi^1_a \right) \varphi^2_x$$

with $D_1 = -i\overline{\varphi^1_a}\, (3\xi^6_5\xi^2_1 + 7) + i\varphi^1_a\, (\xi^6_5\xi^2_1 + 4)$; $D_2 = -i(\overline{\varphi^1_a})^2\, (3\xi^6_5\xi^2_1 + 1) - 8i\overline{\varphi^1_a}\varphi^1_a - 8\overline{\varphi^2_a}\, (\xi^6_5 - \xi^2_1) + 8\varphi^2_a\, (\xi^6_5 + \xi^2_1) + i{\varphi^1_a}^2\, (\xi^6_5\xi^2_1 + 4)$ .

In the case of $J(\xi^3_3)$ in (76) where $\xi^3_3 \neq 0$, the general computations apply with $r = 0, b = 1, \xi^4_3 = \xi^2_1 = -1$. The only difference is that we get now $\chi^3(a,x) = \frac{1}{16} \left( D_1\, (\varphi^1_x)^2 + D_2\, \varphi^1_x \right)$ with $D_1 = -4i\overline{\varphi^1_a} + 3i\varphi^1_a$; $D_2 = (2i - \xi^3_3)\, (\overline{\varphi^1_a})^2 + 4(\xi^3_3 - 2i)\, \overline{\varphi^1_a}\varphi^1_a - 8(i\xi^3_3 + 2)\, \overline{\varphi^2_a} + 8i\xi^3_3\, \varphi^2_a + (3i - \xi^3_3)\, {\varphi^1_a}^2$.

## 11 Lie Algebra $M14_\gamma (\gamma = \pm 1)$.

Commutation relations for $M14_\gamma : [x_1, x_3] = x_4$; $[x_1, x_4] = x_6$; $[x_2, x_3] = x_5$; $[x_2, x_5] = \gamma x_6$. $M14_{-1}$ has no CS. We consider the case $M14_1$.

$$J = \begin{pmatrix} 0 & -\frac{1}{\xi^2_1} & 0 & 0 & 0 & 0 \\ \boxed{\xi^2_1} & 0 & 0 & 0 & 0 & 0 \\ \boxed{\xi^3_1} & \boxed{\xi^3_2} & * & -(\xi^6_6\xi^3_5 - \xi^6_5\xi^3_6)\xi^2_1 & \boxed{\xi^3_5} & \boxed{\xi^3_6} \\ \boxed{\xi^4_1} & * & * & -\frac{(\xi^6_6\xi^3_5 - \xi^6_5\xi^3_6)^2}{\xi^3_6} & -\frac{((\xi^6_5\xi^3_5+1)\xi^3_6 - \xi^6_6{\xi^3_5}^2)\xi^2_1}{\xi^3_6} & (\xi^6_6\xi^3_5 - \xi^6_5\xi^3_6)\xi^2_1 \\ \boxed{\xi^5_1} & * & * & -\frac{((\xi^6_5\xi^3_5-1)\xi^3_6 - \xi^6_6{\xi^3_5}^2)\xi^2_1}{\xi^3_6} & -\frac{{\xi^3_5}^2}{\xi^3_6} & -\xi^3_5 \\ * & * & * & -\frac{({\xi^6_6}^2\xi^3_5 - \xi^6_6\xi^6_5\xi^3_6 + \xi^3_5)\xi^2_1}{\xi^3_6} & \boxed{\xi^6_5} & \boxed{\xi^6_6} \end{pmatrix} \qquad (78)$$

where $\mathbf{J^3_3} = ({\xi^6_5}^2{\xi^3_6}^2 + {\xi^3_5}^2 + {\xi^6_6}^2{\xi^3_5}^2 - (2\xi^6_5\xi^3_5 + 1)\xi^6_6\xi^3_6)/\xi^3_6$; $\mathbf{J^4_2} = ((\xi^6_6\xi^3_5 - \xi^6_5\xi^3_6)\xi^3_2\xi^2_1 + \xi^5_1\xi^3_6 + \xi^3_5\xi^3_1)/\xi^3_6$; $J^4_3 = ((({\xi^6_6}^2{\xi^3_5}^3 - 3\xi^6_6\xi^6_5\xi^3_6{\xi^3_5}^2 - \xi^6_6\xi^3_6\xi^3_5 + 3{\xi^6_5}^2{\xi^3_6}^2\xi^3_5 + \xi^6_5{\xi^3_6}^2 + {\xi^3_5}^3)\xi^6_6 - ({\xi^6_5}^3{\xi^3_6}^2 + \xi^6_5{\xi^3_5}^2 + \xi^3_5)\xi^3_6)\xi^2_1)/{\xi^3_6}^2$; $\mathbf{J^5_2} = (-(\xi^4_1\xi^3_6 +$

$\xi_5^3\xi_2^3+\xi_5^6\xi_6^3\xi_1^3\xi_1^2-\xi_6^6\xi_5^3\xi_1^3\xi_1^2))/\xi_6^3$; $\mathbf{J_3^5} = (-((\xi_6^6\xi_5^3-2\xi_5^6\xi_6^3)\xi_6^6{\xi_5^3}^2+{\xi_5^6}^2{\xi_6^3}^2\xi_5^3-\xi_5^6{\xi_6^3}^2+{\xi_5^3}^3))/{\xi_6^3}^2$; $\mathbf{J_1^6} = (-(\xi_6^3\xi_2^3\xi_1^2+{\xi_5^3}^2\xi_1^3+\xi_1^5\xi_6^3\xi_5^3+(\xi_5^6\xi_1^3+\xi_1^4\xi_1^2)\xi_5^6{\xi_6^3}^2+((\xi_6^6\xi_5^3-2\xi_5^6\xi_6^3)\xi_5^3\xi_1^3-(\xi_1^4\xi_5^3\xi_1^2+\xi_1^3)\xi_6^3)\xi_6^6))/{\xi_6^3}^2$; $\mathbf{J_2^6} = (\xi_6^6\xi_1^5\xi_5^3+\xi_6^6\xi_2^3\xi_1^2-\xi_5^6\xi_1^5\xi_6^3+\xi_1^4\xi_5^3\xi_1^2+\xi_1^3)/(\xi_6^3\xi_1^2)$; $\mathbf{J_3^6} = ({\xi_6^6}^3{\xi_5^3}^2-2{\xi_6^6}^2\xi_5^6\xi_6^3\xi_5^3-{\xi_6^6}^2\xi_6^3+\xi_6^6{\xi_5^6}^2{\xi_6^3}^2+\xi_6^6{\xi_5^3}^2-\xi_6^3)/{\xi_6^3}^2$; and the parameters are subject to the condition

$$\xi_1^2 = \pm 1\ ;\ \xi_6^3 \neq 0. \tag{79}$$

Now the automorphism group of $M14_1$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b_1^1 & b_1^2 u & 0 & 0 & 0 & 0 \\ b_1^2 & -b_1^1 u & 0 & 0 & 0 & 0 \\ 0 & 0 & b_3^3 & 0 & 0 & 0 \\ b_1^4 & b_2^4 & b_3^4 & b_3^3 b_1^1 & b_3^3 b_1^2 u & 0 \\ -(b_2^4 - b_1^2 k)u & b_1^4 u - b_1^1 k & b_3^5 & b_3^3 b_1^2 & -b_3^3 b_1^1 u & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_3^5 b_1^2 + b_3^4 b_1^1 & -(b_3^5 b_1^1 - b_3^4 b_1^2)u & ({b_1^2}^2 + {b_1^1}^2) b_3^3 \end{pmatrix}$$

where $({b_1^2}^2 + {b_1^1}^2) b_3^3 \neq 0$ and $u = \pm 1,\ k \in \mathbb{R}$. Taking suitable values for the $b_j^i$'s, equivalence by $\Phi$ leads to the case where $\xi_1^3 = \xi_2^3 = \xi_5^3 = \xi_1^4 = \xi_1^5 = \xi_6^6 = 0$. Then equivalence by

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\xi_1^2/2 & 0 & -\xi_5^6\xi_1^2 & 1 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & \xi_1^2 & 0 \\ \xi_5^6/2 & 0 & 0 & -\xi_5^6\xi_1^2 & 0 & 1/|\xi_6^3| \end{pmatrix}$$

leads to the case where moreover $\xi_5^6 = 0, \xi_1^2 = 1, {\xi_6^3}^2 = 1$ :

$$J(\xi_6^3) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_6^3 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1/\xi_6^3 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \xi_6^3 = \pm 1. \tag{80}$$

The 2 matrices corresponding to $\xi_6^3 = \pm 1$ are not equivalent. Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_4$; $[\tilde{x}_1, \tilde{x}_6] = -\xi_6^3\tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_6] = \xi_6^3\tilde{x}_4$.

From (78), $\mathfrak{X}_{M14_1}$ is a submanifold of dimension 8 in $\mathbb{R}^{36}$. There are only 2 orbits, and any CS on $M14_1$ is equivalent to one of the two non equivalent structures in (80).

## 11.1

$$X_1 = \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^3} \quad , \quad X_2 = \frac{\partial}{\partial y^1} - x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial y^3}.$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J(\xi_6^3)$ in (80), where $\xi_6^3 = \pm 1$. Then $H_{\mathbb{C}}(G) = \{f \in C^\infty(G_0)\ ;\ \tilde{X}_j^- f = 0\ \forall j = 1,3,5\}$. One has

$$\tilde{X}_1^- = 2\frac{\partial}{\partial \overline{w^1}} - x^2 \frac{\partial}{\partial y^2} - ix^2 \frac{\partial}{\partial x^3} - iw^3 \frac{\partial}{\partial y^3} \quad , \quad \tilde{X}_3^- = 2\frac{\partial}{\partial \overline{w^2}} \quad , \quad \tilde{X}_5^- = 2\frac{\partial}{\partial \overline{w^3}}$$

where $w^1 = x^1 + iy^1 \quad , \quad w^2 = x^2 - i\xi_6^3\, y^3 \quad , \quad w^3 = x^3 - iy^2$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$2\frac{\partial f}{\partial \overline{w^1}} - \xi_6^3 w^3 \frac{\partial f}{\partial w^2} = 0.$$

The 3 functions $\varphi^1 = w^1 \quad , \quad \varphi^2 = w^2 + \frac{\xi_6^3}{2} w^3 \overline{w^1} \quad , \quad \varphi^3 = w^3$ are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a global chart on $G$. We determine now how the multiplication

of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned} w^1_{ax} &= w^1_a + w^1_x \\ w^2_{ax} &= w^2_a + w^2_x - i\xi^3_6(-b^2 x^1 + \frac{1}{2}\, a^2 {x^1}^2 - a^3 y^1 + \frac{1}{2}\, a^2 {y^1}^2) \\ w^3_{ax} &= w^3_a + w^3_x - a^2 y^1 + i a^2 x^1. \end{aligned}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x + \chi^2(a,x) \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi^3(a,x)$$

where

$$\begin{aligned} \chi^2(a,x) &= \frac{1}{8}\, \varphi^1_x \left( 2i\xi^3_6 \overline{\varphi^1_a}(\overline{\varphi^2_a} + \varphi^2_a) + 4\xi^3_6 \overline{\varphi^3_a} - i(\overline{\varphi^1_a})^2 \varphi^3_a - i\, \overline{\varphi^1_a \varphi^3_a} \varphi^1_a \right) + \frac{\xi^3_6}{2}\, \overline{\varphi^1_a} \varphi^3_x; \\ \chi^3(a,x) &= \frac{i}{4}\, \varphi^1_x \left( -3\xi^3_6(\overline{\varphi^1_a}\varphi^3_a + \varphi^1_a \overline{\varphi^3_a}) + 2(\varphi^2_a + \overline{\varphi^2_a}) \right). \end{aligned}$$

## 12 Lie Algebra $M18_\gamma(\gamma = \pm 1)$.

Commutation relations for $M18_\gamma$ : $[x_1, x_2] = x_3$; $[x_1, x_3] = x_4$; $[x_1, x_4] = x_6$; $[x_2, x_3] = x_5$; $[x_2, x_5] = \gamma x_6$. $M18_{-1}$ has no CS. We consider the case $M18_1$. Then $J$ is the same matrix as (78) and the parameters are subject to the same condition (79). This comes as no surprise, since the commutations relations of $M18_\gamma$ are simply those of $M14_\gamma$ plus $[x_1, x_2] = x_3$, and any $J \in \mathfrak{X}_{M18_1}$ has $\xi^1_k = \xi^2_k = 0$ for $3 \leqslant k \leqslant 6$ and $\xi^1_1 = \xi^2_2 = 0$.

Now the automorphism group of $M18_1$ is comprised of the matrices

$$\Phi = \begin{pmatrix} b^1_1 & b^2_1 u & 0 & 0 & 0 & 0 \\ b^2_1 & -b^1_1 u & 0 & 0 & 0 & 0 \\ b^3_1 & b^3_2 & -({b^2_1}^2 + {b^1_1}^2)u & 0 & 0 & 0 \\ b^4_1 & b^4_2 & b^3_2 b^1_1 - b^3_1 b^2_1 u & -({b^2_1}^2 + {b^1_1}^2) b^1_1 u & -({b^2_1}^2 + {b^1_1}^2) b^2_1 & 0 \\ b^5_1 & b^5_2 & b^3_2 b^2_1 + b^3_1 b^1_1 u & -({b^2_1}^2 + {b^1_1}^2) b^2_1 u & ({b^2_1}^2 + {b^1_1}^2) b^1_1 & 0 \\ b^6_1 & b^6_2 & b^4_2 b^1_1 - b^4_1 b^2_1 u + b^5_1 b^1_1 u + b^5_2 b^2_1 & ({b^2_1}^2 + {b^1_1}^2) b^3_2 & -({b^2_1}^2 + {b^1_1}^2) b^3_1 & -({b^2_1}^2 + {b^1_1}^2)^2 u \end{pmatrix}$$

where ${b^2_1}^2 + {b^1_1}^2 \neq 0$ and $u = \pm 1$. Taking suitable values for the $b^i_j$'s, equivalence by $\Phi$ leads to the case where $\xi^3_1 = 0, \xi^3_2 = 0, \xi^3_5 = 0, \xi^4_1 = 0, \xi^5_1 = 0, \xi^6_5 = 0, \xi^6_6 = 0$. Then equivalence by $\Phi = \mathrm{diag}(1, \xi^2_1, \xi^2_1/|\xi^3_6|, \xi^2_1/|\xi^3_6|, 1/|\xi^3_6|, \xi^2_1/|\xi^3_6|^2)$ leads to the case where moreover $\xi^2_1 = 1, {\xi^3_6}^2 = 1$, that is the same matrix $J(\xi^3_6)$ as in (80) with the same condition. Again, the two matrices corresponding to $\xi^3_6 = \pm 1$ are not equivalent. Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_3] = \tilde{x}_4$; $[\tilde{x}_1, \tilde{x}_6] = -\xi^3_6 \tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_3] = \tilde{x}_5$; $[\tilde{x}_2, \tilde{x}_6] = \xi^3_6 \tilde{x}_4$. From (78), $\mathfrak{X}_{M18_1}$ is a submanifold of dimension 8 in $\mathbb{R}^{36}$. There are only 2 orbits, and any $J \in \mathfrak{X}_{M18_1}$ is equivalent to one of the two non equivalent structures in (80).

### 12.1

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} - y^1\, \frac{\partial}{\partial x^2} - x^2\, \frac{\partial}{\partial y^2} + \frac{1}{2}\, {y^1}^2\, \frac{\partial}{\partial x^3} - \left( y^2 + \frac{1}{6}\, {y^1}^3 \right) \frac{\partial}{\partial y^3} \\ X_2 &= \frac{\partial}{\partial y^1} - x^2\, \frac{\partial}{\partial x^3} - x^3\, \frac{\partial}{\partial y^3}. \end{aligned}$$

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J(\xi^3_6)$ in (80), where $\xi^3_6 = \pm 1$. Then $H_{\mathbb{C}}(G) = \{ f \in C^\infty(G_0) \; ; \; \tilde{X}^-_j\, f = 0 \; \forall j = 1, 3, 5 \}$. One has

$$\begin{aligned} \tilde{X}^-_1 &= 2\, \frac{\partial}{\partial \overline{w^1}} - y^1\, \frac{\partial}{\partial x^2} - x^2\, \frac{\partial}{\partial y^2} + \left( \frac{1}{2}\, {y^1}^2 - i x^2 \right) \frac{\partial}{\partial x^3} - \left( y^2 + \frac{1}{6}\, {y^1}^3 + i x^3 \right) \frac{\partial}{\partial y^3} \quad , \\ \tilde{X}^-_3 &= 2 \frac{\partial}{\partial \overline{w^2}} \quad , \tilde{X}^-_5 = 2\, \frac{\partial}{\partial \overline{w^3}} \quad , \end{aligned}$$

where $w^1 = x^1 + iy^1 \quad , \quad w^2 = x^2 - i\xi_6^3\, y^3 \quad , \quad w^3 = x^3 - iy^2$. Then $f \in C^\infty(G_0)$ is in $H_{\mathbb{C}}(G)$ if and only if it is holomorphic with respect to $w^2$ and $w^3$ and satisfies the equation

$$2\,\frac{\partial f}{\partial \overline{w^1}} + \left(-y^1 + \frac{i\xi_6^3}{6}\,{y^1}^3 - \xi_6^3 w^3\right)\,\frac{\partial f}{\partial w^2} + \frac{1}{2}\,{y^1}^2\,\frac{\partial f}{\partial w^3} = 0.$$

The 3 functions

$$\begin{aligned}
\varphi^1 &= w^1 \\
\varphi^2 &= w^2 + \frac{\xi_6^3}{2}\,w^3\overline{w^1} + \frac{1}{8}\,(w^1 - \overline{w^1})^2\left(i - \frac{\xi_6^3}{48}\,(w^1 - \overline{w^1})^2\right) + \frac{\xi_6^3}{384}\,(\overline{w^1})^2\left(6{w^1}^2 - 8w^1\overline{w^1} + 3(\overline{w^1})^2\right) \\
\varphi^3 &= w^3 + \frac{1}{48}\,\overline{w^1}\left(3{w^1}^2 - 3w^1\overline{w^1} + (\overline{w^1})^2\right)
\end{aligned}$$

are holomorphic. Let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$. $F$ is a a global chart on $G$. We determine now how the multiplication of $G$ looks like in that chart. Let $a, x \in G$ with respective second kind canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (1). With obvious notations, computations yield:

$$\begin{aligned}
w^1_{ax} &= w^1_a + w^1_x \\
w^2_{ax} &= w^2_a + w^2_x \\
&\quad -b^1x^1 + i\xi_6^3\left(a^3y^1 + b^2x^1 - \frac{1}{2}\,a^2{x^1}^2 + \frac{1}{6}\,{b^1}^3x^1 + \frac{1}{6}\,b^1{x^1}^3 + \frac{1}{2}\,{b^1}^2x^1y^1 - \frac{1}{2}\,{y^1}^2(a^2 - b^1x^1)\right) \\
w^3_{ax} &= w^3_a + w^3_x + \frac{1}{2}\,{b^1}^2x^1 - y^1(a^2 - b^1x^1) - i\left(\frac{1}{2}\,b^1{x^1}^2 - a^2x^1\right).
\end{aligned}$$

We then get

$$\varphi^1_{ax} = \varphi^1_a + \varphi^1_x \quad , \quad \varphi^2_{ax} = \varphi^2_a + \varphi^2_x + \chi^2(a,x) \quad , \quad \varphi^3_{ax} = \varphi^3_a + \varphi^3_x + \chi^3(a,x)$$

where

$$\begin{aligned}
\chi^2(a,x) &= \frac{\xi_6^3}{32}\,(\varphi^1_x)^3\left(\overline{\varphi^1_a} - \varphi^1_a\right) + \frac{\xi_6^3}{64}\,(\varphi^1_x)^2\left(4(\overline{\varphi^1_a})^2 - 3(\varphi^1_a)^2\right) + \frac{1}{512}\,D_2\,\varphi^1_x + \frac{\xi_6^3}{2}\,\varphi^3_x\overline{\varphi^1_a}; \\
\chi^3(a,x) &= \frac{1}{16}\,(\varphi^1_x)^2\left(4\overline{\varphi^1_a} - 3\varphi^1_a\right) + \frac{1}{256}D_3\,\varphi^1_x;
\end{aligned}$$

where

$$\begin{aligned}
D_2 = 16\xi_6^3\Big(&-(\overline{\varphi^1_a})^3 + 8i\overline{\varphi^1_a\varphi^2_a} + 2\overline{\varphi^1_a}(\varphi^1_a)^2 + 8i\overline{\varphi^1_a}\varphi^2_a + 16\overline{\varphi^3_a} - (\varphi^1_a)^3\Big) \\
&+ i\Big((\overline{\varphi^1_a})^5 - 4(\overline{\varphi^1_a})^4\varphi^1_a + 8(\overline{\varphi^1_a})^3(\varphi^1_a)^2 - 4(\overline{\varphi^1_a})^2(\varphi^1_a)^3 \\
&\qquad -64(\overline{\varphi^1_a})^2\varphi^3_a - 64\overline{\varphi^1_a\varphi^3_a}\varphi^1_a + \overline{\varphi^1_a}(\varphi^1_a)^4 - 256(\overline{\varphi^1_a} - \varphi^1_a)\Big)
\end{aligned}$$

$$\begin{aligned}
D_3 = i\xi_6^3\Big(&(\overline{\varphi^1_a})^4 - 4(\overline{\varphi^1_a})^3\varphi^1_a + 8(\overline{\varphi^1_a})^2(\varphi^1_a)^2 - 4\overline{\varphi^1_a}(\varphi^1_a)^3 - 64\overline{\varphi^1_a}\varphi^3_a - 64\overline{\varphi^3_a}\varphi^1_a + (\varphi^1_a)^4\Big) \\
&- 32(\overline{\varphi^1_a})^2 - 16(\varphi^1_a)^2 + 64\overline{\varphi^1_a}\varphi^1_a + 128i(\overline{\varphi^2_a} + \varphi^2_a).
\end{aligned}$$

## 13 Lie Algebra $M5$.

Commutation relations for $M5 : [x_1, x_3] = x_5;\ [x_1, x_4] = x_6;\ [x_2, x_3] = -x_6;\ [x_2, x_4] = x_5$. This is the realification of the 3-dimensional complex Heisenberg Lie algebra $\mathfrak{n}\ \ [Z_1, Z_2] = Z_3$ we get by letting $x_1 = Z_1, x_2 = -iZ_1, x_3 = Z_2, x_4 = iZ_2, x_5 = Z_3, x_6 = iZ_3$.

### 13.1 Case ${\xi_4^2}^2 + {\xi_3^2}^2 \neq 0$.

$$J = \begin{pmatrix} \boxed{\xi_1^1} & * & \boxed{\xi_3^1} & \boxed{\xi_4^1} & 0 & 0 \\ \boxed{\xi_1^2} & * & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix} \tag{81}$$

where $\mathbf{J_2^1} = (-(((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2)\xi_4^1+(\xi_3^2-2\xi_4^1)\xi_5^5\xi_3^1)\xi_5^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^1-(({\xi_4^1}^2+{\xi_3^1}^2)\xi_1^2-\xi_3^2\xi_3^1\xi_1^1-\xi_4^2\xi_4^1\xi_1^1)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$; $\mathbf{J_2^2} = (-(((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2-\xi_5^5\xi_3^1)\xi_4^2+(\xi_3^2-\xi_4^1)\xi_5^5\xi_3^2)\xi_5^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^2+((\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)\xi_3^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$; $\mathbf{J_1^3} = (-(((\xi_1^2\xi_3^1-\xi_4^1\xi_1^1-\xi_3^2\xi_1^1)\xi_1^2+({\xi_1^1}^2+1)\xi_4^2)\xi_5^6+(({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_1^2))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$; $\mathbf{J_2^3} = (-(((({\xi_5^6}^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_3^2\xi_4^1+{\xi_5^6}^2{\xi_4^1}^2-2\xi_5^6\xi_5^5\xi_4^2\xi_3^2+2\xi_5^6\xi_5^5\xi_4^2\xi_4^1-2\xi_5^6\xi_5^5\xi_3^2\xi_3^1+2\xi_5^6\xi_5^5\xi_4^1\xi_3^1+\xi_5^6\xi_4^2\xi_1^2\xi_3^1-\xi_5^6\xi_3^2\xi_1^2\xi_4^1-2\xi_5^6\xi_3^2\xi_3^1\xi_1^1+{\xi_5^5}^2{\xi_4^2}^2+2{\xi_5^5}^2\xi_4^2\xi_3^1-2{\xi_5^5}^2{\xi_3^2}^2+2{\xi_5^5}^2\xi_3^2\xi_4^1+{\xi_5^5}^2{\xi_3^1}^2-\xi_5^5{\xi_4^2}^2\xi_1^1+\xi_5^5\xi_4^2\xi_1^2\xi_4^1-2\xi_5^5{\xi_3^2}^2\xi_1^1+2\xi_5^5\xi_3^2\xi_1^2\xi_3^1+\xi_5^5{\xi_3^1}^2\xi_1^1+{\xi_4^2}^2+2\xi_4^2\xi_3^1-2{\xi_3^2}^2+2\xi_3^2\xi_4^1+{\xi_3^1}^2+(\xi_1^2\xi_3^1+2\xi_4^1\xi_1^1)\xi_5^6\xi_3^1)\xi_4^2+((\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5-2{\xi_5^5}^2\xi_3^1-2\xi_3^1)(\xi_3^2-\xi_4^1)\xi_3^2-(2{\xi_3^2}^2\xi_1^1-2\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1)\xi_5^5\xi_3^1+(2{\xi_3^2}^3\xi_1^1-2{\xi_3^2}^2\xi_1^2\xi_3^1-2{\xi_3^2}^2\xi_4^1\xi_1^1+3\xi_3^2\xi_1^2\xi_4^1\xi_3^1-2\xi_1^2{\xi_4^1}^2\xi_3^1)\xi_5^6)\xi_5^5-({\xi_4^2}^2\xi_1^1-\xi_4^2\xi_1^2\xi_4^1-\xi_4^2\xi_3^1\xi_1^1+2{\xi_3^2}^2\xi_1^1-2\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1)(\xi_4^2+\xi_3^1))\xi_5^6+({\xi_5^5}^2+1)^2(\xi_4^2+\xi_3^1)^2\xi_3^2+((\xi_3^2\xi_1^1-\xi_1^2\xi_3^1-\xi_4^1\xi_1^1)\xi_4^2+\xi_1^2{\xi_4^1}^2)(\xi_3^2-\xi_4^1){\xi_5^6}^3+(((\xi_4^2{\xi_1^1}^2-\xi_4^2-\xi_1^2\xi_4^1\xi_1^1-({\xi_1^1}^2+1)\xi_3^1)\xi_3^2-(\xi_1^2\xi_3^1\xi_1^1+\xi_4^1{\xi_1^1}^2-\xi_4^1)\xi_4^2+(\xi_1^2\xi_4^1\xi_3^1+2{\xi_4^1}^2\xi_1^1+{\xi_3^1}^2\xi_1^1)\xi_1^2)\xi_4^2+({\xi_3^2}^2{\xi_1^1}^2-2\xi_3^2\xi_1^2\xi_3^1\xi_1^1+{\xi_1^2}^2{\xi_3^1}^2+\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_4^1}^2)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_4^1){\xi_5^6}^2))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)^2{\xi_5^6}^2)$;
$\mathbf{J_3^3} = (-((({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_3^2-({\xi_3^2}^2\xi_1^1-\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1-\xi_4^2\xi_3^1\xi_1^1)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$;
$J_4^3 = (-((({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)-(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5)\xi_4^2-(\xi_4^2\xi_3^2\xi_1^1-\xi_4^2\xi_1^2\xi_3^1-\xi_4^2\xi_4^1\xi_1^1+\xi_1^2{\xi_4^1}^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\xi_5^6)$; $\mathbf{J_1^4} = -((\xi_1^2\xi_4^1+\xi_3^1\xi_1^1)\xi_1^2-({\xi_1^1}^2+1)\xi_3^2-\xi_4^2\xi_1^2\xi_1^1+(\xi_4^2+\xi_3^1)\xi_5^5\xi_1^2-(\xi_3^2-\xi_4^1)\xi_5^6\xi_1^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$; $\mathbf{J_2^4} = (-((((\xi_3^2-\xi_4^1)\xi_5^6-2\xi_5^5\xi_3^1)(\xi_3^2-\xi_4^1)\xi_5^5\xi_3^2+({\xi_5^6}^2{\xi_3^2}^2-2{\xi_5^6}^2\xi_3^2\xi_4^1+{\xi_5^6}^2{\xi_4^1}^2-2\xi_5^6\xi_5^5\xi_4^2\xi_3^2+2\xi_5^6\xi_5^5\xi_4^2\xi_4^1-2\xi_5^6\xi_5^5\xi_3^2\xi_3^1+2\xi_5^6\xi_5^5\xi_4^1\xi_3^1+{\xi_5^5}^2{\xi_4^2}^2+2{\xi_5^5}^2\xi_4^2\xi_3^1-2{\xi_5^5}^2{\xi_3^2}^2+2{\xi_5^5}^2\xi_3^2\xi_4^1+{\xi_5^5}^2{\xi_3^1}^2-2\xi_5^5{\xi_4^2}^2\xi_1^1+2\xi_5^5\xi_4^2\xi_1^2\xi_4^1-2\xi_5^5\xi_4^2\xi_3^1\xi_1^1+\xi_5^5\xi_3^2\xi_1^2\xi_3^1-2\xi_5^5\xi_3^2\xi_4^1\xi_1^1+3\xi_5^5\xi_1^2\xi_4^1\xi_3^1)\xi_4^2-({\xi_3^2}^2\xi_1^2-\xi_3^2\xi_1^2\xi_4^1+2\xi_3^2\xi_3^1\xi_1^1-2\xi_1^2{\xi_3^1}^2)\xi_5^5\xi_4^1)\xi_5^6+({\xi_5^5}^2+1)(\xi_5^5\xi_4^2\xi_3^2+\xi_5^5\xi_3^2\xi_3^1-\xi_4^2\xi_3^2\xi_1^1+\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2)(\xi_4^2+\xi_3^1)+(2(\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)(\xi_3^2+\xi_4^1))(\xi_3^2-\xi_4^1){\xi_5^6}^2-(((\xi_3^2\xi_4^1\xi_1^1-\xi_1^2\xi_4^1\xi_3^1+{\xi_4^1}^2\xi_1^1+2{\xi_3^1}^2\xi_1^1)\xi_1^2-(\xi_1^1+1)(\xi_1^1-1)\xi_3^2\xi_3^1)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_3^1-({\xi_4^2}^2{\xi_1^1}^2-2\xi_4^2\xi_1^2\xi_4^1\xi_1^1+{\xi_3^2}^2{\xi_1^1}^2-{\xi_3^2}^2-\xi_3^2\xi_1^2\xi_3^1\xi_1^1+\xi_3^2\xi_4^1{\xi_1^1}^2+\xi_3^2\xi_4^1+{\xi_1^2}^2{\xi_4^1}^2-\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_3^1}^2)\xi_4^2)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)^2\xi_5^6)$; $\mathbf{J_3^4} = -(\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2-\xi_4^2\xi_3^2\xi_1^1+(\xi_4^2+\xi_3^1)\xi_5^5\xi_3^2-(\xi_3^2-\xi_4^1)\xi_5^6\xi_3^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$; $\mathbf{J_4^4} = ((\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)\xi_4^1-(\xi_4^2+\xi_3^1)\xi_5^5\xi_4^2+(\xi_3^2-\xi_4^1)\xi_5^6\xi_4^2)/(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)$;
$\mathbf{J_1^5} = ((((\xi_5^5-\xi_1^1)\xi_1^6-\xi_2^6\xi_1^2)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)-(\xi_3^2-\xi_4^1)\xi_3^6\xi_5^5\xi_1^2+((\xi_1^2\xi_3^1-\xi_4^1\xi_1^1-\xi_3^2\xi_1^1)\xi_1^2+({\xi_1^1}^2+1)\xi_4^2)\xi_3^6)\xi_5^6+((\xi_1^2\xi_4^1+\xi_3^1\xi_1^1)\xi_1^2-({\xi_1^1}^2+1)\xi_3^2-\xi_4^2\xi_1^2\xi_1^1)\xi_5^6\xi_4^6-((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2-\xi_5^5\xi_3^1)\xi_5^6\xi_4^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^6)\xi_1^2)/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1){\xi_5^6}^2)$; $\mathbf{J_2^5} = ((((\xi_3^2-\xi_4^1)\xi_5^6-2\xi_5^5\xi_3^1)(\xi_3^2-\xi_4^1)\xi_5^6\xi_5^5\xi_3^2+{\xi_5^6}^3\xi_4^2{\xi_3^2}^2-2{\xi_5^6}^3\xi_4^2\xi_3^2\xi_4^1+{\xi_5^6}^3\xi_4^2{\xi_4^1}^2-2{\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_3^2+2{\xi_5^6}^2\xi_5^5{\xi_4^2}^2\xi_4^1-2{\xi_5^6}^2\xi_5^5\xi_4^2\xi_3^2\xi_3^1+2{\xi_5^6}^2\xi_5^5\xi_4^2\xi_4^1\xi_3^1+\xi_5^6{\xi_5^5}^2{\xi_4^2}^3+2\xi_5^6{\xi_5^5}^2{\xi_4^2}^2\xi_3^1-2\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^2}^2+2\xi_5^6{\xi_5^5}^2\xi_4^2\xi_3^2\xi_4^1+\xi_5^6{\xi_5^5}^2\xi_4^2{\xi_3^1}^2-2\xi_5^6\xi_5^5{\xi_4^2}^3\xi_1^1+2\xi_5^6\xi_5^5{\xi_4^2}^2\xi_1^2\xi_4^1-2\xi_5^6\xi_5^5{\xi_4^2}^2\xi_3^1\xi_1^1+\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_1^2\xi_3^1-2\xi_5^6\xi_5^5\xi_4^2\xi_3^2\xi_4^1\xi_1^1+3\xi_5^6\xi_5^5\xi_4^2\xi_1^2\xi_4^1\xi_3^1+{\xi_5^5}^3{\xi_4^2}^2\xi_3^2+2{\xi_5^5}^3\xi_4^2\xi_3^2\xi_3^1+{\xi_5^5}^3\xi_3^2{\xi_3^1}^2-{\xi_5^5}^2{\xi_4^2}^2\xi_3^2\xi_1^1+{\xi_5^5}^2\xi_4^2\xi_3^2\xi_1^2\xi_4^1-2{\xi_5^5}^2\xi_4^2\xi_3^2\xi_3^1\xi_1^1+{\xi_5^5}^2\xi_4^2\xi_1^2{\xi_3^1}^2+\xi_5^5{\xi_4^2}^2\xi_3^2+2\xi_5^5\xi_4^2\xi_3^2\xi_3^1+\xi_5^5\xi_3^2{\xi_3^1}^2+(\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2){\xi_5^5}^2\xi_3^1-({\xi_3^2}^2\xi_1^2-\xi_3^2\xi_1^2\xi_4^1+2\xi_3^2\xi_3^1\xi_1^1-2\xi_1^2{\xi_3^1}^2)\xi_5^6\xi_5^5\xi_4^1-(\xi_4^2\xi_3^2\xi_1^1-\xi_3^2\xi_1^2\xi_4^1+\xi_3^2\xi_3^1\xi_1^1-\xi_1^2{\xi_3^1}^2)(\xi_4^2+\xi_3^1))\xi_5^6\xi_4^6+({\xi_5^5}^2+1)^2(\xi_4^2+\xi_3^1)^2\xi_3^6\xi_3^2+(2(\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)(\xi_3^2+\xi_4^1))(\xi_3^2-\xi_4^1){\xi_5^6}^3\xi_4^6-((\xi_2^6\xi_3^2+\xi_1^6\xi_3^1)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)-(\xi_5^5{\xi_4^2}^2+\xi_5^5\xi_4^2\xi_3^1-2\xi_5^5{\xi_3^2}^2+2\xi_5^5\xi_3^2\xi_4^1-{\xi_4^2}^2\xi_1^1+\xi_4^2\xi_1^2\xi_4^1+\xi_4^2\xi_3^1\xi_1^1-2{\xi_3^2}^2\xi_1^1+2\xi_3^2\xi_1^2\xi_3^1-\xi_1^2\xi_4^1\xi_3^1)\xi_3^6)({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_5^6+((\xi_2^6\xi_4^2+\xi_1^6\xi_4^1)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)+(\xi_3^2-\xi_4^1)\xi_3^6\xi_5^5\xi_4^2+((\xi_3^2\xi_1^1-\xi_1^2\xi_3^1-\xi_4^1\xi_1^1)\xi_4^2+\xi_1^2{\xi_4^1}^2)\xi_3^6)(\xi_3^2-\xi_4^1){\xi_5^6}^3-((((\xi_3^2\xi_4^1\xi_1^1-\xi_1^2\xi_4^1\xi_3^1+{\xi_4^1}^2\xi_1^1+2{\xi_3^1}^2\xi_1^1)\xi_1^2-(\xi_1^1+1)(\xi_1^1-1)\xi_3^2\xi_3^1)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_3^1-({\xi_4^2}^2{\xi_1^1}^2-2\xi_4^2\xi_1^2\xi_4^1\xi_1^1+{\xi_3^2}^2{\xi_1^1}^2-{\xi_3^2}^2-\xi_3^2\xi_1^2\xi_3^1\xi_1^1+\xi_3^2\xi_4^1{\xi_1^1}^2+\xi_3^2\xi_4^1+{\xi_1^2}^2{\xi_4^1}^2-\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_3^1}^2)\xi_4^2){\xi_5^6}^2\xi_4^6-(((2\xi_5^5\xi_4^2\xi_3^2-2\xi_5^5\xi_4^2\xi_4^1+2\xi_5^5\xi_3^2\xi_3^1-2\xi_5^5\xi_4^1\xi_3^1-\xi_4^2\xi_1^2\xi_3^1+\xi_3^2\xi_1^2\xi_4^1+2\xi_3^2\xi_3^1\xi_1^1-(\xi_1^2\xi_3^1+2\xi_4^1\xi_1^1)\xi_3^1)\xi_4^2-(\xi_3^2-\xi_4^1)^2\xi_5^5\xi_3^2-(2{\xi_3^2}^3\xi_1^1-2{\xi_3^2}^2\xi_1^2\xi_3^1-2{\xi_3^2}^2\xi_4^1\xi_1^1+3\xi_3^2\xi_1^2\xi_4^1\xi_3^1-2\xi_1^2{\xi_4^1}^2\xi_3^1))\xi_3^6\xi_5^5-(({\xi_4^2}^2\xi_1^1-\xi_4^2\xi_1^2\xi_4^1+{\xi_3^2}^2\xi_1^1-\xi_3^2\xi_1^2\xi_3^1-({\xi_4^2}^2-{\xi_3^2}^2+2\xi_3^2\xi_4^1)\xi_5^5)\xi_2^6+(\xi_4^2\xi_4^1\xi_1^1+\xi_3^2\xi_3^1\xi_1^1-\xi_1^2{\xi_4^1}^2-\xi_1^2{\xi_3^1}^2-(\xi_4^2\xi_4^1-\xi_3^2\xi_3^1+2\xi_4^1\xi_3^1)\xi_5^5)\xi_1^6)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)-(((\xi_4^2{\xi_1^1}^2-\xi_4^2-\xi_1^2\xi_4^1\xi_1^1-({\xi_1^1}^2+1)\xi_3^1)\xi_3^2-(\xi_1^2\xi_3^1\xi_1^1+\xi_4^1{\xi_1^1}^2-\xi_4^1)\xi_4^2+(\xi_1^2\xi_4^1\xi_3^1+2{\xi_4^1}^2\xi_1^1+{\xi_3^1}^2\xi_1^1)\xi_1^2)\xi_4^2+({\xi_3^2}^2{\xi_1^1}^2-2\xi_3^2\xi_1^2\xi_3^1\xi_1^1+{\xi_1^2}^2{\xi_3^1}^2+\xi_1^2\xi_4^1\xi_3^1\xi_1^1+{\xi_4^1}^2)\xi_3^2-({\xi_4^1}^2+{\xi_3^1}^2){\xi_1^2}^2\xi_4^1)\xi_3^6){\xi_5^6}^2)/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)^2{\xi_5^6}^3)$; $\mathbf{J_3^5} = (-((((\xi_5^6\xi_3^2-\xi_5^6\xi_4^1-\xi_5^5\xi_4^2-\xi_5^5\xi_3^1)\xi_5^6\xi_4^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^6)\xi_3^2-(\xi_3^2\xi_1^2\xi_4^1-\xi_3^2\xi_3^1\xi_1^1+\xi_1^2{\xi_3^1}^2-\xi_4^2\xi_3^2\xi_1^1)\xi_5^6\xi_4^6+((\xi_2^6\xi_3^2+\xi_1^6\xi_3^1)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)-(\xi_4^2\xi_3^1-{\xi_3^2}^2)\xi_3^6\xi_5^5+({\xi_3^2}^2\xi_1^1-\xi_3^2\xi_1^2\xi_3^1+\xi_1^2\xi_4^1\xi_3^1-\xi_4^2\xi_3^1\xi_1^1)\xi_3^6)\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1){\xi_5^6}^2)$;
$\mathbf{J_4^5} = (-((\xi_5^6\xi_4^2\xi_3^2-\xi_5^6\xi_4^2\xi_4^1-\xi_5^5{\xi_4^2}^2-2\xi_5^5\xi_4^2\xi_3^1+\xi_5^5\xi_3^2\xi_4^1)\xi_5^6\xi_4^6-({\xi_5^5}^2+1)(\xi_4^2+\xi_3^1)\xi_3^6\xi_4^2+((\xi_4^2\xi_1^1-\xi_1^2\xi_4^1)\xi_4^2+(\xi_3^2\xi_1^1-\xi_1^2\xi_3^1)\xi_4^1)\xi_5^6\xi_4^6+((\xi_4^2\xi_3^2\xi_1^1-\xi_4^2\xi_1^2\xi_3^1-\xi_4^2\xi_4^1\xi_1^1+\xi_1^2{\xi_4^1}^2+(\xi_3^2-\xi_4^1)\xi_5^5\xi_4^2)\xi_3^6+(\xi_2^6\xi_4^2+\xi_1^6\xi_4^1)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1))\xi_5^6))/((\xi_4^2\xi_3^1-\xi_3^2\xi_4^1){\xi_5^6}^2)$; and the parameters are subject to the condition $\xi_5^6({\xi_4^2}^2+{\xi_3^2}^2)(\xi_4^2\xi_3^1-\xi_3^2\xi_4^1)\neq 0$.

## 13.2 Case $\xi_1^2 \neq 0$.

### 13.2.1 Case $\xi_1^2\xi_4^2 \neq 0$.

$$J = \begin{pmatrix} * & * & * & * & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & \boxed{\xi_4^2} & 0 & 0 \\ * & * & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix}$$

where the parameters are subject to the condition

$$\xi_1^2\xi_5^6\xi_4^2 C_2 \neq 0 \tag{82}$$

with $C_2 = ((\xi_5^6\xi_4^3\xi_4^2 + \xi_5^6\xi_4^2\xi_1^2 + \xi_5^5\xi_4^3\xi_3^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_3^2 + (\xi_4^2\xi_2^2 + \xi_3^2\xi_1^2 + \xi_3^3\xi_4^2)\xi_5^6\xi_5^5)\xi_4^2 - ((\xi_3^3\xi_4^2\xi_2^2 - \xi_3^3\xi_3^2\xi_1^2 - \xi_4^2)\xi_4^2 - (\xi_4^2\xi_2^2 - \xi_3^2\xi_1^2)\xi_4^3\xi_3^2)\xi_5^6$. The starred $J_j^i$'s are smooth rational functions under condition (82). However we do not give them here, as some are huge and we do not have explicit use of them in the rest of the paper. We refer instead to ([7], pp. 133-136).

### 13.2.2 Case $\xi_1^2\xi_3^2 \neq 0, \xi_4^2 = 0$.

$$J = \begin{pmatrix} * & * & -\frac{(\xi_3^3+\xi_2^2)\xi_3^2}{\xi_1^2} & -\frac{\xi_4^3\xi_3^2}{\xi_1^2} & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & \boxed{\xi_3^2} & 0 & 0 & 0 \\ * & * & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & \boxed{\xi_5^5} & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & \boxed{\xi_5^6} & -\xi_5^5 \end{pmatrix}$$

where $\mathbf{J_1^1} = (-(({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2) + \xi_5^6\xi_2^2\xi_1^2 + (\xi_4^3\xi_2^2+\xi_3^3\xi_1^2+(\xi_4^3+\xi_1^2)\xi_5^5)\xi_5^6))/(\xi_5^6\xi_1^2)$; $\mathbf{J_2^1} = (-((\xi_3^3+\xi_2^2)\xi_2^2 - \xi_4^3\xi_1^2 - (\xi_3^3+\xi_2^2)\xi_5^5 - (\xi_4^3+\xi_1^2)\xi_5^6))/\xi_1^2$; $\mathbf{J_1^3} = ((\xi_4^3\xi_2^2+\xi_3^3\xi_1^2+(\xi_4^3+\xi_1^2)\xi_5^5)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2))/(\xi_5^6\xi_3^2)$; $\mathbf{J_2^3} = (-(\xi_5^6\xi_4^3+\xi_5^6\xi_1^2+\xi_5^5\xi_3^3+\xi_5^5\xi_2^2+\xi_4^3\xi_1^2-\xi_3^3\xi_2^2+1))/\xi_3^2$; $\mathbf{J_1^4} = ((((\xi_3^3+\xi_2^2)\xi_5^5+2\xi_5^6\xi_1^2+2{\xi_5^5}^2+2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2 + {\xi_5^6}^2\xi_4^3{\xi_1^2}^2 + {\xi_5^6}^2{\xi_1^2}^3 + \xi_5^6{\xi_5^5}^2{\xi_4^3}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_1^2 + \xi_5^6{\xi_5^5}^2{\xi_1^2}^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_1^2 + 3\xi_5^6\xi_5^5\xi_4^3\xi_2^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^3\xi_3^3 + 2{\xi_5^5}^3\xi_4^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3{\xi_2^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2 + ((\xi_3^3+\xi_2^2)\xi_1^2+2\xi_4^3\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^6 + ({\xi_4^3}^2{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_2^2\xi_1^2 + {\xi_1^2}^2 + ({\xi_2^2}^2+{\xi_1^2}^2)\xi_4^3\xi_1^2){\xi_5^6}^2)/({\xi_5^6}^2\xi_4^3\xi_3^2\xi_1^2)$;
$J_2^4 = (-({\xi_5^6}^2\xi_5^5{\xi_4^3}^2 + 2{\xi_5^6}^2\xi_5^5\xi_4^3\xi_1^2 + {\xi_5^6}^2\xi_5^5{\xi_1^2}^2 + {\xi_5^6}^2{\xi_4^3}^2\xi_2^2 - {\xi_5^6}^2\xi_2^2{\xi_1^2}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_3^3 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_2^2 + 2\xi_5^6{\xi_5^5}^2\xi_3^3\xi_1^2 + 2\xi_5^6{\xi_5^5}^2\xi_2^2\xi_1^2 + \xi_5^6\xi_5^5{\xi_4^3}^2\xi_1^2 + \xi_5^6\xi_5^5\xi_4^3{\xi_1^2}^2 - 2\xi_5^6\xi_5^5\xi_3^3\xi_2^2\xi_1^2 - 2\xi_5^6\xi_5^5{\xi_2^2}^2\xi_1^2 + \xi_5^6{\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_5^6\xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_5^6\xi_4^3\xi_3^3 - \xi_5^6\xi_4^3{\xi_2^2}^3 - \xi_5^6\xi_4^3\xi_2^2{\xi_1^2}^2 + \xi_5^6\xi_4^3\xi_2^2 + {\xi_5^5}^3{\xi_3^3}^2 + 2{\xi_5^5}^3\xi_3^3\xi_2^2 + {\xi_5^5}^3{\xi_2^2}^2 + {\xi_5^5}^2\xi_4^3\xi_3^3\xi_1^2 + {\xi_5^5}^2\xi_4^3\xi_2^2\xi_1^2 - {\xi_5^5}^2{\xi_3^3}^2\xi_2^2 - 2{\xi_5^5}^2\xi_3^3{\xi_2^2}^2 - {\xi_5^5}^2{\xi_2^2}^3 + \xi_5^5{\xi_3^3}^2 + 2\xi_5^5\xi_3^3\xi_2^2 + \xi_5^5{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_1^2 + \xi_4^3\xi_2^2\xi_1^2 - {\xi_3^3}^2\xi_2^2 - 2\xi_3^3{\xi_2^2}^2 - {\xi_2^2}^3))/(\xi_5^6\xi_4^3\xi_3^2\xi_1^2)$; $\mathbf{J_3^4} = ((\xi_5^6\xi_4^3\xi_1^2 + \xi_5^6{\xi_1^2}^2 + \xi_5^5\xi_4^3\xi_3^3 + \xi_5^5\xi_4^3\xi_2^2 + 2(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)^2 + ({\xi_2^2}^2 + {\xi_1^2}^2 + \xi_3^3\xi_2^2)\xi_5^6\xi_4^3)/(\xi_5^6\xi_4^3\xi_1^2)$; $\mathbf{J_4^4} = (({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2) + (\xi_5^5\xi_4^3+\xi_5^5\xi_1^2+\xi_4^3\xi_2^2)\xi_5^6)/(\xi_5^6\xi_1^2)$; $\mathbf{J_1^5} = (-(((((\xi_2^6\xi_1^2 - \xi_1^6\xi_5^5)\xi_5^6\xi_3^2 + ({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2)\xi_3^6)\xi_1^2 + (\xi_4^3\xi_2^2+\xi_3^3\xi_1^2+(\xi_4^3+\xi_1^2)\xi_5^5)(\xi_3^6\xi_1^2 - \xi_1^6\xi_3^2)\xi_5^6)\xi_4^3 + ({\xi_4^3}^2{\xi_2^2}^2 + \xi_4^3\xi_3^3\xi_2^2\xi_1^2 + {\xi_1^2}^2 + ({\xi_2^2}^2+{\xi_1^2}^2)\xi_4^3\xi_1^2)\xi_5^6\xi_4^6 - (({\xi_5^5}^2+1)(\xi_3^3+\xi_2^2) + \xi_5^6\xi_2^2\xi_1^2)\xi_1^6\xi_4^3\xi_3^2)\xi_5^6 + ((((\xi_3^3+\xi_2^2)\xi_5^5 + 2\xi_5^6\xi_1^2 + 2{\xi_5^5}^2+2)(\xi_3^3+\xi_2^2)\xi_5^5\xi_1^2 + {\xi_5^6}^2\xi_4^3{\xi_1^2}^2 + {\xi_5^6}^2{\xi_1^2}^3 + \xi_5^6{\xi_5^5}^2{\xi_4^3}^2 + 2\xi_5^6{\xi_5^5}^2\xi_4^3\xi_1^2 + \xi_5^6{\xi_5^5}^2{\xi_1^2}^2 + 2\xi_5^6\xi_5^5{\xi_4^3}^2\xi_2^2 + \xi_5^6\xi_5^5\xi_4^3\xi_3^3\xi_1^2 + 3\xi_5^6\xi_5^5\xi_4^3\xi_2^2\xi_1^2 + 2{\xi_5^5}^3\xi_4^3\xi_3^3 + 2{\xi_5^5}^3\xi_4^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3\xi_3^3\xi_2^2 + 2{\xi_5^5}^2\xi_4^3{\xi_2^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)^2(\xi_3^3+\xi_2^2)^2 + ((\xi_3^3+\xi_2^2)\xi_1^2 + 2\xi_4^3\xi_2^2)(\xi_3^3+\xi_2^2)\xi_5^6)\xi_4^6))/({\xi_5^6}^3\xi_4^3\xi_3^2\xi_1^2)$;
$J_2^5 = ((((\xi_5^6{\xi_4^3}^2 + 2\xi_5^6\xi_4^3\xi_1^2 + \xi_5^6{\xi_1^2}^2 + 2\xi_5^5\xi_4^3\xi_3^3 + 2\xi_5^5\xi_4^3\xi_2^2 + {\xi_4^3}^2\xi_1^2 + \xi_4^3{\xi_1^2}^2 + 2(\xi_5^5-\xi_2^2)(\xi_3^3+\xi_2^2)\xi_1^2)\xi_5^5 + (\xi_4^3+\xi_1^2)(\xi_4^3-\xi_1^2)\xi_5^6\xi_2^2)\xi_5^6 + ({\xi_5^5}^2+1)(\xi_5^5\xi_3^3 + \xi_5^5\xi_2^2 + \xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 - {\xi_2^2}^2)(\xi_3^3+\xi_2^2))\xi_4^6 + (\xi_3^6\xi_1^2 - \xi_1^6\xi_3^2)(\xi_4^3+\xi_1^2){\xi_5^6}^2\xi_4^3 + (\xi_4^3\xi_2^2\xi_1^2 - \xi_3^3{\xi_2^2}^2 + \xi_3^3 - ({\xi_2^2}^2 + {\xi_1^2}^2 - 1)\xi_2^2)\xi_5^6\xi_4^6\xi_4^3 - (((\xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 - {\xi_2^2}^2 + (\xi_3^3+\xi_2^2)\xi_5^5)\xi_1^6 - (\xi_5^5 - \xi_2^2)\xi_2^6\xi_1^2)\xi_3^2 - (\xi_4^3\xi_1^2 - \xi_3^3\xi_2^2 + 1 + (\xi_3^3+\xi_2^2)\xi_5^5)\xi_3^6\xi_1^2)\xi_5^6\xi_4^3)/({\xi_5^6}^2\xi_4^3\xi_3^2\xi_1^2)$;
$\mathbf{J_3^5} = (-({\xi_5^6}^2\xi_4^6\xi_4^3\xi_1^2 + {\xi_5^6}^2\xi_4^6{\xi_1^2}^2 + \xi_5^6\xi_4^6\xi_5^5\xi_4^3\xi_3^3 + \xi_5^6\xi_4^6\xi_5^5\xi_4^3\xi_2^2 + 2\xi_5^6\xi_4^6\xi_5^5\xi_3^3\xi_1^2 + 2\xi_5^6\xi_4^6\xi_5^5\xi_2^2\xi_1^2 + \xi_5^6\xi_4^6\xi_4^3\xi_3^3\xi_2^2 + \xi_5^6\xi_4^6\xi_4^3{\xi_2^2}^2 + \xi_5^6\xi_4^6\xi_4^3{\xi_1^2}^2 - \xi_5^6\xi_3^6\xi_5^5\xi_4^3\xi_1^2 + \xi_5^6\xi_3^6\xi_4^3\xi_3^3\xi_1^2 + \xi_5^6\xi_2^6\xi_4^3\xi_3^2\xi_1^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^3\xi_3^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^2\xi_2^2 + \xi_4^6{\xi_5^5}^2{\xi_3^3}^2 + 2\xi_4^6{\xi_5^5}^2\xi_3^3\xi_2^2 + \xi_4^6{\xi_5^5}^2{\xi_2^2}^2 + \xi_4^6{\xi_3^3}^2 + 2\xi_4^6\xi_3^3\xi_2^2 + \xi_4^6{\xi_2^2}^2))/({\xi_5^6}^2\xi_4^3\xi_1^2)$; $\mathbf{J_4^5} = (-(\xi_5^6\xi_4^6\xi_5^5\xi_4^3 + \xi_5^6\xi_4^6\xi_4^3\xi_2^2 + \xi_5^6\xi_3^6\xi_4^3\xi_1^2 - \xi_5^6\xi_1^6\xi_4^3\xi_3^3 + \xi_4^6{\xi_5^5}^2\xi_3^3 + \xi_4^6{\xi_5^5}^2\xi_2^2 + \xi_4^6\xi_3^3 + \xi_4^6\xi_2^2))/({\xi_5^6}^2\xi_1^2)$; and the parameters are subject to the conditions $\xi_1^2\xi_5^6\xi_3^2\xi_4^3 \neq 0$.

### 13.2.3 Case $\xi_1^2 \neq 0, \xi_3^2 = \xi_4^2 = 0, \xi_2^2 + \xi_3^3 \neq 0$.

$$J = \begin{pmatrix} -\xi_2^2 & -\frac{{\xi_2^2}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & -\frac{\xi_1^4\xi_4^3-\xi_3^3\xi_1^3+\xi_1^3\xi_2^2}{\xi_1^2} & \boxed{\xi_3^3} & \boxed{\xi_4^3} & 0 & 0 \\ \boxed{\xi_1^4} & \frac{\xi_1^4\xi_4^3\xi_3^3+\xi_1^4\xi_4^3\xi_2^2+{\xi_3^3}^2\xi_1^3+\xi_1^3}{\xi_4^3\xi_1^2} & -\frac{{\xi_3^3}^2+1}{\xi_4^3} & -\xi_3^3 & 0 & 0 \\ * & * & * & * & * & * \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & * & * \end{pmatrix}$$

where $\mathbf{J^5_1} = (\xi_4^6\xi_1^4{\xi_4^3}^2\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_2^2}^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_1^2}^2 + \xi_4^6\xi_1^4\xi_4^3 + \xi_4^6\xi_1^4{\xi_3^3}^2\xi_1^2 + \xi_4^6\xi_1^4\xi_1^2 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3\xi_1^3{\xi_2^2}^2 + \xi_3^6\xi_4^3\xi_1^3{\xi_1^2}^2 + \xi_3^6\xi_4^3\xi_1^3 + \xi_3^6{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_3^6\xi_1^3\xi_1^2 + \xi_2^6{\xi_4^3}^2{\xi_1^2}^2 + \xi_2^6\xi_4^3{\xi_2^2}^2\xi_1^2 + \xi_2^6\xi_4^3{\xi_1^2}^3 + \xi_2^6\xi_4^3\xi_1^2 + \xi_2^6{\xi_3^3}^2{\xi_1^2}^2 + \xi_2^6{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_1^6\xi_4^3\xi_3^3{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_3^3 - \xi_1^6\xi_4^3{\xi_2^2}^3 - \xi_1^6\xi_4^3\xi_2^2{\xi_1^2}^2 - \xi_1^6\xi_4^3\xi_2^2 - 2\xi_1^6{\xi_3^3}^2\xi_2^2\xi_1^2 - 2\xi_1^6\xi_2^2\xi_1^2)/({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2 + 1)({\xi_3^3}^2 + 1))$; $\mathbf{J^5_2} = (\xi_4^6\xi_1^4{\xi_4^3}^3\xi_3^3\xi_1^2 + \xi_4^6\xi_1^4{\xi_4^3}^3\xi_2^2\xi_1^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3{\xi_2^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3{\xi_1^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_3^3 + \xi_4^6\xi_1^4{\xi_4^3}^2{\xi_2^2}^3 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_2^2{\xi_1^2}^2 + \xi_4^6\xi_1^4{\xi_4^3}^2\xi_2^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_3^3}^3\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3{\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3\xi_3^3\xi_1^2 + \xi_4^6\xi_1^4\xi_4^3\xi_2^2\xi_1^2 + \xi_4^6{\xi_4^3}^2{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_4^6{\xi_4^3}^2\xi_1^3\xi_1^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3{\xi_2^2}^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3{\xi_1^2}^2 + \xi_4^6\xi_4^3{\xi_3^3}^2\xi_1^3 + \xi_4^6\xi_4^3\xi_1^3{\xi_2^2}^2 + \xi_4^6\xi_4^3\xi_1^3{\xi_1^2}^2 + \xi_4^6\xi_4^3\xi_1^3 + \xi_4^6{\xi_3^3}^4\xi_1^3\xi_1^2 + 2\xi_4^6{\xi_3^3}^2\xi_1^3\xi_1^2 + \xi_4^6\xi_1^3\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^4\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^3{\xi_2^2}^2 - \xi_3^6\xi_1^4{\xi_4^3}^3{\xi_1^2}^2 - \xi_3^6\xi_1^4{\xi_4^3}^3 - \xi_3^6\xi_1^4{\xi_4^3}^2{\xi_3^3}^2\xi_1^2 - \xi_3^6\xi_1^4{\xi_4^3}^2\xi_1^2 - \xi_3^6{\xi_4^3}^3\xi_3^3\xi_1^3\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_1^3\xi_2^2\xi_1^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3{\xi_2^2}^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3{\xi_1^2}^2 - \xi_3^6{\xi_4^3}^2\xi_3^3\xi_1^3 + \xi_3^6{\xi_4^3}^2\xi_1^3{\xi_2^2}^3 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_2^2{\xi_1^2}^2 + \xi_3^6{\xi_4^3}^2\xi_1^3\xi_2^2 - \xi_3^6\xi_4^3{\xi_3^3}^3\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3{\xi_3^3}^2\xi_1^3\xi_2^2\xi_1^2 - \xi_3^6\xi_4^3\xi_3^3\xi_1^3\xi_1^2 + \xi_3^6\xi_4^3\xi_1^3\xi_2^2\xi_1^2 + 2\xi_2^6{\xi_4^3}^3\xi_2^2{\xi_1^2}^2 - \xi_2^6{\xi_4^3}^2\xi_3^3{\xi_2^2}^2\xi_1^2 + \xi_2^6{\xi_4^3}^2\xi_3^3{\xi_1^2}^3 - \xi_2^6{\xi_4^3}^2\xi_3^3\xi_1^2 + \xi_2^6{\xi_4^3}^2{\xi_2^2}^3\xi_1^2 + \xi_2^6{\xi_4^3}^2\xi_2^2{\xi_1^2}^3 + \xi_2^6{\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_1^6{\xi_4^3}^3{\xi_2^2}^2\xi_1^2 - \xi_1^6{\xi_4^3}^3\xi_1^2 - \xi_1^6{\xi_4^3}^2{\xi_2^2}^4 - \xi_1^6{\xi_4^3}^2{\xi_2^2}^2{\xi_1^2}^2 - 2\xi_1^6{\xi_4^3}^2{\xi_2^2}^2 - \xi_1^6{\xi_4^3}^2{\xi_1^2}^2 - \xi_1^6{\xi_4^3}^2 - \xi_1^6\xi_4^3{\xi_3^3}^2{\xi_2^2}^2\xi_1^2 - \xi_1^6\xi_4^3{\xi_3^3}^2\xi_1^2 - \xi_1^6\xi_4^3{\xi_2^2}^2\xi_1^2 - \xi_1^6\xi_4^3\xi_1^2)/(\xi_4^3\xi_1^2({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2 + 1)({\xi_3^3}^2 + 1)))$; $\mathbf{J^5_3} = (-\xi_4^6{\xi_4^3}^2{\xi_3^3}^2\xi_1^2 - \xi_4^6{\xi_4^3}^2\xi_1^2 - \xi_4^6\xi_4^3{\xi_3^3}^2{\xi_2^2}^2 - \xi_4^6\xi_4^3{\xi_3^3}^2{\xi_1^2}^2 - \xi_4^6\xi_4^3{\xi_3^3}^2 - \xi_4^6\xi_4^3{\xi_2^2}^2 - \xi_4^6\xi_4^3{\xi_1^2}^2 - \xi_4^6\xi_4^3 - \xi_4^6{\xi_3^3}^4\xi_1^2 - 2\xi_4^6{\xi_3^3}^2\xi_1^2 - \xi_4^6\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_3^3\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_2^2\xi_1^2 + 2\xi_3^6{\xi_4^3}^2\xi_3^3{\xi_1^2}^2 + \xi_3^6\xi_4^3{\xi_3^3}^3\xi_1^2 - \xi_3^6\xi_4^3{\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_3^6\xi_4^3\xi_3^3\xi_1^2 - \xi_3^6\xi_4^3\xi_2^2\xi_1^2)/(\xi_4^3({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2 + 1)({\xi_3^3}^2 + 1)))$; $\mathbf{J^5_4} = (-\xi_4^6{\xi_4^3}^2\xi_3^3\xi_1^2 + \xi_4^6{\xi_4^3}^2\xi_2^2\xi_1^2 - 2\xi_4^6\xi_4^3\xi_3^3{\xi_2^2}^2 - 2\xi_4^6\xi_4^3\xi_3^3 - \xi_4^6{\xi_3^3}^3\xi_1^2 - \xi_4^6{\xi_3^3}^2\xi_2^2\xi_1^2 - \xi_4^6\xi_3^3\xi_1^2 - \xi_4^6\xi_2^2\xi_1^2 + \xi_3^6{\xi_4^3}^3\xi_1^2 + \xi_3^6{\xi_4^3}^2{\xi_2^2}^2 + \xi_3^6{\xi_4^3}^2{\xi_1^2}^2 + \xi_3^6{\xi_4^3}^2 + \xi_3^6\xi_4^3{\xi_3^3}^2\xi_1^2 + \xi_3^6\xi_4^3\xi_1^2)/({\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2 + 1)({\xi_3^3}^2 + 1))$; $\mathbf{J^5_5} = (-{\xi_4^3}^2\xi_2^2\xi_1^2 + \xi_4^3\xi_3^3{\xi_2^2}^2 - \xi_4^3\xi_3^3{\xi_1^2}^2 + \xi_4^3\xi_3^3 + {\xi_3^3}^2\xi_2^2\xi_1^2 + \xi_2^2\xi_1^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2 + 1)\xi_1^2)$; $\mathbf{J^5_6} = ({\xi_4^3}^2{\xi_2^2}^2 + {\xi_4^3}^2 + 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 + 2\xi_4^3\xi_1^2 + {\xi_3^3}^2{\xi_1^2}^2 + {\xi_1^2}^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2 + 1)\xi_1^2)$; $\mathbf{J^6_5} = (-{\xi_4^3}^2{\xi_1^2}^2 + 2\xi_4^3\xi_3^3\xi_2^2\xi_1^2 - 2\xi_4^3\xi_1^2 - {\xi_3^3}^2{\xi_2^2}^2 - {\xi_3^3}^2 - {\xi_2^2}^2 - 1)/({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2 + 1)\xi_1^2)$; $\mathbf{J^6_6} = ({\xi_4^3}^2\xi_2^2\xi_1^2 - \xi_4^3\xi_3^3{\xi_2^2}^2 + \xi_4^3\xi_3^3{\xi_1^2}^2 - \xi_4^3\xi_3^3 - {\xi_3^3}^2\xi_2^2\xi_1^2 - \xi_2^2\xi_1^2)/({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2 + 1)\xi_1^2)$; and the parameters are subject to the condition

$$\xi_1^2\xi_4^3(\xi_3^3 + \xi_2^2)({\xi_4^3}^2\xi_1^2 + \xi_4^3({\xi_2^2}^2 + {\xi_1^2}^2 + 1) + ({\xi_3^3}^2 + 1)\xi_1^2) \neq 0. \tag{83}$$

(Note that ${\xi_4^3}^2{\xi_1^2}^2 - 2\xi_4^3\xi_1^2(\xi_3^3\xi_2^2 - 1) + ({\xi_2^2}^2 + 1)({\xi_3^3}^2 + 1) \neq 0$ is automatic from $\xi_3^3 + \xi_2^2 \neq 0$.)

### 13.2.4 Case $\xi_1^2 \neq 0, \xi_3^2 = \xi_4^2 = 0, \xi_3^3 = -\xi_2^2, \xi_4^3 = -\xi_1^2$.

$$J = \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ \xi_1^3 & \xi_1^4 & 0 & -\xi_1^2 & 0 & 0 \\ \xi_1^4 & -\xi_1^3 & \frac{1}{\xi_1^2} & 0 & 0 & 0 \\ \frac{-\xi_4^6\xi_1^4-\xi_3^6\xi_1^3-\xi_2^6\xi_1^2+\xi_1^6\xi_5^5}{\xi_5^6} & \frac{\xi_4^6\xi_1^3\xi_1^2-\xi_3^6\xi_1^4\xi_1^2+\xi_2^6\xi_5^5\xi_1^2+\xi_1^6}{\xi_5^6\xi_1^2} & \frac{-\xi_4^6+\xi_3^6\xi_5^5\xi_1^2}{\xi_5^6\xi_1^2} & \frac{\xi_4^6\xi_5^5+\xi_3^6\xi_1^2}{\xi_5^6} & \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & \xi_5^6 & -\xi_5^5 \end{pmatrix}$$

where $\xi_1^2 = \pm 1$.

### 13.2.5 Case $\xi_1^2 \neq 0, \xi_3^2 = \xi_4^2 = 0, \xi_3^3 = -\xi_2^2, \xi_4^3 \neq -\xi_1^2$.

$$J = \begin{pmatrix} -\xi_2^2 & -\frac{{\xi_2^2}^2+1}{\xi_1^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^2} & \boxed{\xi_2^2} & 0 & 0 & 0 & 0 \\ \boxed{\xi_1^3} & -\frac{\xi_1^4\xi_4^3-2\xi_1^3\xi_2^2}{\xi_1^2} & -\xi_2^2 & \boxed{\xi_4^3} & 0 & 0 \\ \boxed{\xi_1^4} & \frac{\xi_1^3({\xi_2^2}^2+1)}{\xi_4^3\xi_1^2)} & -\frac{{\xi_2^2}^2+1}{\xi_4^3} & \xi_2^2 & 0 & 0 \\ * & * & * & * & \frac{\xi_2^2(-\xi_4^3+\xi_1^2)}{\xi_4^3+\xi_1^2} & \frac{{\xi_4^3}^2{\xi_2^2}^2+{\xi_4^3}^2-2\xi_4^3{\xi_2^2}^2\xi_1^2+2\xi_4^3\xi_1^2+{\xi_2^2}^2{\xi_1^2}^2+{\xi_1^2}^2}{{\xi_4^3}^2\xi_1^2+\xi_4^3{\xi_2^2}^2+\xi_4^3{\xi_1^2}^2+\xi_4^3+{\xi_2^2}^2\xi_1^2+\xi_1^2} \\ \boxed{\xi_1^6} & \boxed{\xi_2^6} & \boxed{\xi_3^6} & \boxed{\xi_4^6} & -\frac{\xi_4^3\xi_1^2+{\xi_2^2}^2+1}{\xi_4^3+\xi_1^2} & \frac{\xi_2^2(\xi_4^3-\xi_1^2)}{\xi_4^3+\xi_1^2} \end{pmatrix}$$

where $\mathbf{J^5_1} = (\xi^6_4\xi^4_1\xi^3_4 + \xi^6_4\xi^4_1\xi^2_1 + \xi^6_3\xi^3_4\xi^3_1 + \xi^6_3\xi^3_1\xi^2_1 + \xi^6_2\xi^3_4\xi^2_1 + \xi^6_2{\xi^2_1}^2 - 2\xi^6_1\xi^2_2\xi^2_1)/(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1)$; $\mathbf{J^5_2} = (\xi^6_4\xi^3_4\xi^3_1{\xi^2_2}^2 + \xi^6_4\xi^3_4\xi^3_1 + \xi^6_4\xi^3_1{\xi^2_2}^2\xi^2_1 + \xi^6_4\xi^3_1\xi^2_1 - \xi^6_3\xi^4_1{\xi^3_4}^3 - \xi^6_3\xi^4_1{\xi^3_4}^2\xi^2_1 + 2\xi^6_3{\xi^3_4}^2\xi^3_1\xi^2_2 + 2\xi^6_3\xi^3_4\xi^3_1\xi^2_2\xi^2_1 + 2\xi^6_2{\xi^3_4}^2\xi^2_2\xi^2_1 - \xi^6_1{\xi^3_4}^2{\xi^2_2}^2 - \xi^6_1{\xi^3_4}^2 - \xi^6_1\xi^3_4{\xi^2_2}^2\xi^2_1 - \xi^6_1\xi^3_4\xi^2_1)/(\xi^3_4\xi^2_1(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1))$; $\mathbf{J^5_3} = (-\xi^6_4\xi^3_4{\xi^2_2}^2 - \xi^6_4\xi^3_4 - \xi^6_4{\xi^2_2}^2\xi^2_1 - \xi^6_4\xi^2_1 - 2\xi^6_3\xi^3_4\xi^2_2\xi^2_1)/(\xi^3_4(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1))$; $\mathbf{J^5_4} = (\xi^3_4(2\xi^6_4\xi^2_2 + \xi^6_3\xi^3_4 + \xi^6_3\xi^2_1))/(\xi^3_4\xi^2_1 + {\xi^2_2}^2 + 1)$; and the parameters are subject to the condition $\xi^2_1\xi^3_4(\xi^3_4 + \xi^2_1)({\xi^2_2}^2 + \xi^3_4\xi^2_1 + 1) \neq 0$.

### 13.2.6 Conclusions for the case $\xi^2_1 \neq 0$.

In each of the 5 subcases in the case $\xi^2_1 \neq 0$, after completing a set of common steps, one is left with solving the two equations 14|5 and 14|6 in the 14 variables $\xi^2_1, \xi^2_2, \xi^2_3, \xi^2_4, \xi^3_1, \xi^3_3, \xi^3_4, \xi^4_1, \xi^5_5, \xi^6_1, \xi^6_2, \xi^6_3, \xi^6_4, \xi^6_5$ in the open subset $\xi^6_5(\xi^3_4\xi^2_1 - \xi^3_1\xi^2_4) \neq 0$ of $\mathbb{R}^{14}$. That is, the initial system comprised of all the torsion equations and the equation $J^2 = -1$ in $\mathbb{R}^{36}$ is reduced after the common steps to a system equivalent to the 2 mentioned equations, which reads

$$\begin{cases} f = 0 \\ g = 0 \end{cases} \tag{84}$$

where : $f = \xi^6_5\xi^5_5\xi^3_4{\xi^2_1}^2 + \xi^6_5\xi^5_5{\xi^2_1}^3 + \xi^6_5\xi^4_1\xi^3_4\xi^2_4\xi^2_1 - \xi^6_5\xi^4_1\xi^3_1{\xi^2_4}^2 + \xi^6_5\xi^3_4\xi^2_2{\xi^2_1}^2 + \xi^6_5\xi^3_3\xi^3_1\xi^2_4\xi^2_1 + \xi^6_5\xi^3_3{\xi^2_1}^3 - \xi^6_5{\xi^3_1}^2\xi^2_4\xi^2_3 - \xi^6_5\xi^3_1\xi^2_4\xi^2_2\xi^2_1 - \xi^6_5\xi^3_1\xi^2_3{\xi^2_1}^2 + {\xi^5_5}^2\xi^3_3{\xi^2_1}^2 + {\xi^5_5}^2\xi^2_2{\xi^2_1}^2 + \xi^3_3{\xi^2_1}^2 + \xi^2_2{\xi^2_1}^2$ and $g = \xi^6_5\xi^3_4{\xi^2_1}^2 + \xi^6_5{\xi^2_1}^3 + \xi^5_5\xi^3_3{\xi^2_1}^2 + \xi^5_5\xi^2_2{\xi^2_1}^2 - \xi^4_1\xi^3_4\xi^2_3\xi^2_1 + \xi^4_1\xi^3_1\xi^2_4\xi^2_3 + \xi^3_4{\xi^2_1}^3 - \xi^3_3\xi^3_1\xi^2_3\xi^2_1 - \xi^3_3\xi^2_2{\xi^2_1}^2 + {\xi^3_1}^2{\xi^2_3}^2 - \xi^3_1\xi^2_4{\xi^2_1}^2 + \xi^3_1\xi^2_3\xi^2_2\xi^2_1 + {\xi^2_1}^2$. Hence, if $\mathfrak{X}_{\xi^2_1 \neq 0}$ denotes the subset of $\mathfrak{X}_{M5}$ such that $\xi^2_1 \neq 0$, to conclude that $\mathfrak{X}_{\xi^2_1 \neq 0}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$, it will be sufficient to prove that the preceding system is of maximal rank 2, that is in each of the subcases some 2-jacobian doesn't vanish.

• First, one has $\frac{D(f,g)}{D(\xi^4_1,\xi^3_1)} = -\frac{1}{{\xi^2_1}^2}(\xi^3_4\xi^2_1 - \xi^3_1\xi^2_4)({\xi^2_4}^2 + {\xi^2_3}^2)$ hence this 2-jacobian doesn't vanish if $\xi^2_4 \neq 0$ or $\xi^2_3 \neq 0$.

• Suppose $\xi^2_4 = \xi^2_3 = 0$. Then $\frac{D(f,g)}{D(\xi^5_5,\xi^6_5)} = -\frac{1}{{\xi^6_5}^2}\left((\xi^6_5(\xi^3_4 + \xi^2_1) + \xi^5_5(\xi^3_3 + \xi^2_2))^2 + (\xi^3_3 + \xi^2_2)^2\right)$ hence this 2-jacobian doesn't vanish if $\xi^3_3 + \xi^2_2 \neq 0$.

• Suppose $\xi^2_4 = \xi^2_3 = 0$ and $\xi^3_3 = -\xi^2_2$. Then $\frac{D(f,g)}{D(\xi^5_5,\xi^6_5)} = (\xi^3_4 + \xi^2_1)^2$ hence this 2-jacobian doesn't vanish if $\xi^3_4 + \xi^2_1 \neq 0$.

• Suppose $\xi^2_4 = \xi^2_3 = 0$ and $\xi^3_3 = -\xi^2_2$ and $\xi^3_4 = -\xi^2_1$. Then the equation $f = 0$ reads $-2\xi^2_2\xi^2_1 = 0$, hence $\xi^2_2 = 0$. Then the equation $g = 0$ reads ${\xi^2_2}^2 - {\xi^2_1}^2 + 1 = 0$, hence $\xi^2_1 = \pm 1$. Then $\frac{D(f,g)}{D(\xi^2_2,\xi^3_3)} = -2\xi^5_5\xi^2_1$ and $\frac{D(f,g)}{D(\xi^2_3,\xi^2_4)} = {\xi^4_1}^2 + {\xi^3_1}^2$ hence if $\xi^5_5$ or $\xi^4_1$ or $\xi^3_1$ is $\neq 0$, one of these 2-jacobians doesn't vanish. On the other hand, if $\xi^5_5 = \xi^4_1 = \xi^3_1 = 0$, $\frac{D(f,g)}{D(\xi^2_1,\xi^3_3)} = \xi^2_1(1 - {\xi^6_5}^2)$ hence if $\xi^6_5 \neq \pm 1$, this 2-jacobian doesn't vanish. Suppose now that $\xi^6_5 = \pm 1$. Then $\frac{D(f,g)}{D(\xi^2_1,\xi^2_2)} = 2\xi^6_5(\xi^2_1 - \xi^6_5)$ hence if $\xi^6_5 \neq \xi^2_1$, this 2-jacobian doesn't vanish. Suppose finally that $\xi^6_5 = \xi^2_1$. Then $\frac{D(f,g)}{D(\xi^3_3,\xi^3_4)} = 4$. This ends the proof that the system (84) is of maximal rank.

## 13.3

From 13.1, one has that the subset $\mathfrak{X}_{{\xi^2_4}^2 + {\xi^2_3}^2 \neq 0}$ of $\mathfrak{X}_{M5}$ such that ${\xi^2_4}^2 + {\xi^2_3}^2 \neq 0$, is a 12-dimensional submanifold of $\mathbb{R}^{36}$ with the global chart (81). Now if $\xi^2_4 = \xi^2_3 = 0$, then necessarily $\xi^2_1 \neq 0$. Hence

$$\mathfrak{X}_{M5} = \mathfrak{X}_{{\xi^2_4}^2 + {\xi^2_3}^2 \neq 0} \cup \mathfrak{X}_{\xi^2_1 \neq 0}$$

and we conclude from 13.2.6 that $\mathfrak{X}_{M5}$ is a 12-dimensional submanifold of $\mathbb{R}^{36}$. A local chart in a neighborhood of the canonical CS $J_0$ appears in ([7], 6.5, p.155).

## 13.4 Equivalence.

Due to the number of cases that had to be considered in the preceding computations, we'll tackle the equivalence problem in a slightly different way, mixing equations solving and reduction by equivalence. First we give explicit computation of all $\Phi \in \mathrm{Aut}(M5)$.

### 13.4.1

**Lemma 1.** *Aut($M5$) is comprised of all those (real) matrices of the following form :*

$$\Phi = \begin{pmatrix} & & & & 0 & 0 \\ & & \Phi_\dagger & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & H & Ku \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & K & -Hu \end{pmatrix} \tag{85}$$

*with*

$$\Phi_\dagger = \begin{pmatrix} b_1^1 & b_1^2 u & b_3^1 & -b_3^2 u \\ b_1^2 & -b_1^1 u & b_3^2 & b_3^1 u \\ b_1^3 & -b_1^4 u & b_3^3 & b_3^4 u \\ b_1^4 & b_1^3 u & b_3^4 & -b_3^3 u \end{pmatrix} \tag{86}$$

*where* $u = \pm 1$ *and* $H + iK = \det(\mathbf{w}_1, \mathbf{w}_3) \neq 0$ *with* $\mathbf{w_1} = \begin{pmatrix} b_1^1 - ib_1^2 \\ b_1^3 + ib_1^4 \end{pmatrix}$ *and* $\mathbf{w_3} = \begin{pmatrix} b_3^1 - ib_3^2 \\ b_3^3 + ib_3^4 \end{pmatrix}$.

*Proof.* Let $\Phi = (b_j^i) \in \mathrm{Aut}(M5)$. Since $\Phi$ leaves the 2$^{\text{d}}$ central derivative $\mathcal{C}^2(M5)$ invariant, $b_5^i = b_6^i = 0$ for $1 \leqslant i \leqslant 4$. Denote by $ij|k$ the equation obtained by projecting on $x_k$ the equation $[\Phi(x_i), \Phi(x_j)] - \Phi([x_i, x_j]) = 0$. Equations $13|5, 13|6, 14|5, 14|6$ yield $b_5^5 = b_3^4 b_1^2 - b_1^4 b_3^2 + b_3^3 b_1^1 - b_1^3 b_3^1$; $b_5^6 = b_3^4 b_1^1 - b_1^4 b_3^1 - b_3^3 b_1^2 + b_1^3 b_3^2$; $b_6^5 = b_4^4 b_1^2 - b_1^4 b_4^2 + b_4^3 b_1^1 - b_1^3 b_4^1$; $b_6^6 = b_4^4 b_1^1 - b_1^4 b_4^1 - b_4^3 b_1^2 + b_1^3 b_4^2$. Now equations $12|5$ and $12|6$ read respectively $\Delta^{1,3} + \Delta^{2,4} = 0$ and $\Delta^{1,4} - \Delta^{2,3} = 0$, with $\Delta^{i,j}$ the minors formed with the 1$^{\text{st}}$ and 2$^{\text{d}}$ columns and the rows indicated by the indices in the matrix $\Phi_\dagger = \left(b_j^i\right)_{1 \leqslant i,j \leqslant 4}$. If we introduce for $1 \leqslant j \leqslant 4$ $\quad \mathbf{w}_j = \begin{pmatrix} b_j^1 - ib_j^2 \\ b_j^3 + ib_j^4 \end{pmatrix}$, then $\det(\mathbf{w}_1, \mathbf{w}_2) = \Delta^{1,3} + \Delta^{2,4} + i(\Delta^{1,4} - \Delta^{2,3})$, hence equations $12|5$ and $12|6$ are equivalent to the single complex equation $\det(\mathbf{w}_1, \mathbf{w}_2) = 0$, i.e. to the existence of $z = \alpha + i\beta \in \mathbb{C}$ such that $\mathbf{w}_2 = z\,\mathbf{w}_1$. In the same way, equations $34|5$ and $34|6$ are equivalent to the existence of $w = \gamma + i\delta \in \mathbb{C}$ such that $\mathbf{w}_4 = w\,\mathbf{w}_3$. No, if we introduce $h = \alpha + \gamma, k = \beta + \delta$, the system $23|5, 23|6$ reads

$$\begin{cases} h\,H - k\,K = 0 \\ k\,H + h\,K = 0 \end{cases} \tag{87}$$

where $H = \Delta'^{2,4} + \Delta'^{1,3},\ K = \Delta'^{1,4} - \Delta'^{2,3}$, $\Delta'^{i,j}$ the minors formed with the 1$^{\text{st}}$ and 3$^{\text{d}}$ columns and the rows indicated by the indices in the matrix $\Phi_\dagger$. Since $H + iK = \det(\mathbf{w}_1, \mathbf{w}_3)$, the case $H = K = 0$ would imply $\det(\mathbf{w}_1, \mathbf{w}_3) = 0$ which in turn leads to $\det \Phi = 0$. Hence $(H, K)$ is a non trivial solution to the system (87). As its determinant is $h^2 + k^2$ we conclude that $h = k = 0$, i.e. $\gamma = -\alpha, \delta = -\beta$. Now we are left only with the system of equations $24|5, 24|6$. It reads

$$\begin{cases} (\alpha^2 - \beta^2 + 1)H - 2\alpha\beta K = 0 \\ 2\alpha\beta H + (\alpha^2 - \beta^2 + 1)K = 0. \end{cases} \tag{88}$$

Again, as $(H, K)$ is a non trivial solution one has $\alpha\beta = 0, \alpha^2 - \beta^2 + 1 = 0$ i.e. $\alpha = 0, \beta = \pm 1$. Then we get (85) with $u = \beta$. $\square$

The subgroup $\mathrm{Aut}(\mathfrak{n}) \subset \mathrm{Aut}(M5)$ of complex automorphisms of the complex Heisenberg Lie algebra $\mathfrak{n}$ is the subgroup comprised of all those matrices in (85) for which $u = -1$ and $b_2^6 = -b_1^5, b_2^5 = b_1^6, b_4^6 = b_3^5, b_4^5 = -b_3^6$.

**13.4.2**

Now, looking for CSs $J$, after completing a set of general steps, one is left to find solutions of the torsion equations and $J^2 = -1$ of the following form:

$$J = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 & 0 & 0 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 & 0 & 0 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_4^3 & 0 & 0 \\ \xi_1^4 & \xi_2^4 & \xi_3^4 & \xi_4^4 & 0 & 0 \\ \xi_1^5 & \xi_2^5 & \xi_3^5 & \xi_4^5 & \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & \xi_5^6 & -\xi_5^5 \end{pmatrix}$$

where the $\xi_j^5$'s are certain linear expressions in the $\xi_k^6$'s ( $1 \leqslant j, k \leqslant 4$) and $\xi_4^4 = -\sum_1^3 \xi_j^j$. Take the following $\Phi \in \mathrm{Aut}(M5)$ :

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ b_1^5 & b_2^5 & b_3^5 & b_4^5 & 1 & 0 \\ b_1^6 & b_2^6 & b_3^6 & b_4^6 & 0 & 1 \end{pmatrix}$$

with $b_j^5 = \frac{1}{\xi_5^6}(-\xi_j^6 + \sum_{k=1}^4 b_k^6 \xi_j^k + \xi_5^5 b_j^6)$. Then equivalence by $\Phi$ leads to the case where $\xi_j^5 = \xi_j^6 = 0 \; \forall j \; 1 \leqslant j \leqslant 4$. Now consider the submatrix

$$J_\dagger = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_4^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_4^3 \\ \xi_1^4 & \xi_2^4 & \xi_3^4 & \xi_4^4 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{89}$$

We get 2 cases 1 and 2 below. Before we proceed further, we record a lemma.

**Lemma 2.** *Let $J$ be a CS on $M5$ of the following form :*

$$J = diag\left(J_\dagger, \begin{pmatrix} \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_5^6 & -\xi_5^5 \end{pmatrix}\right)$$

*where $J_\dagger$ is given in (89).*
*(i) Suppose that $B$ and $C$ are not simultaneously zero. Then $J$ is equivalent to a CS of the same form for which $B \neq 0$.*
*(ii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \neq 0$. Then $J$ is equivalent to a CS of the same form for which $\xi_3^2 = 1, \xi_4^2 = 0$.*
*(iii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix}$. Then $J$ is equivalent to a CS of the same form having the same $B$ and for which $\xi_1^2 = \xi_2^2 = 0$.*
*(iv) Suppose $B = C = 0$ and $\xi_2^2 = -\xi_1^1, \xi_4^4 = -\xi_3^3$. Then $J$ is equivalent to a CS of the same form for which $B = C = 0$ and $\xi_1^1 = \xi_3^3 = 0$.*

*Proof.* (i) The matrix $J$ has the form $\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & E \end{pmatrix}$. Take $\Phi_\dagger = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\Phi = \mathrm{diag}\left(\Phi_\dagger, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. From Lemma 1, $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}.$$

Hence if $B$ and $C$ are not simultaneously zero, one may suppose $B \neq 0$.
(ii) Suppose first $\xi_3^2 = \xi_4^2 = 0$. Then, since $B \neq 0$, the first row of $B$ is not zero. Consider

$\Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ and $\Phi = \mathrm{diag}\left(\Phi_\dagger, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ with $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} UAU & UBU \\ UCU & UDU \end{pmatrix}.$$

where $UBU = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \xi_4^2 & \xi_3^2 \\ \xi_4^1 & \xi_3^1 \end{pmatrix}$. Hence we are led to the case where the second row of $B$ doesn't vanish. Consider that case now. Introduce $\Phi_\dagger = \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$ and $\Phi = \mathrm{diag}\left(\Phi_\dagger, \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\right)$ with $V = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \neq 0$. Then $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} I & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} A & BV \\ V^{-1}C & V^{-1}DV \end{pmatrix}.$$

where

$$BV = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ \xi_3^2 & \xi_4^2 \end{pmatrix} \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} * & * \\ {\xi'}_3^2 & {\xi'}_4^2 \end{pmatrix}$$

with ${\xi'}_3^2 = \alpha\xi_3^2 + \beta\xi_4^2$ , ${\xi'}_4^2 = \alpha\xi_4^2 - \beta\xi_3^2$. We want ${\xi'}_3^2 = 1$ and ${\xi'}_4^2 = 0$. This is a Cramer system in $\alpha, \beta$ since $(\xi_3^2)^2 + (\xi_4^2)^2 \neq 0$, hence it has a nontrivial solution. Hence we are reduced to the case where $\xi_3^2 = 1$ and $\xi_4^2 = 0$.

(iii) Suppose $B = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix}$. Consider $\Phi_\dagger = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix}$ and $\Phi = \mathrm{diag}\left(\Phi_\dagger, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$ with $T = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \neq 0$. Then $\Phi \in \mathrm{Aut}(M5)$. Now

$$\begin{aligned} (\Phi_\dagger)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Phi_\dagger &= \begin{pmatrix} I & 0 \\ -T & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ -TA + C & -TB + D \end{pmatrix} \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \\ &= \begin{pmatrix} A + BT & B \\ * & * \end{pmatrix}. \end{aligned}$$

We want $A + BT = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$, that is $BT = \begin{pmatrix} * & * \\ -\xi_1^2 & -\xi_2^2 \end{pmatrix}$. As $BT = \begin{pmatrix} \xi_3^1 & \xi_4^1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = \begin{pmatrix} * & * \\ \alpha & \beta \end{pmatrix}$, one has to let $\alpha = -\xi_1^2$ and $\beta = -\xi_2^2$, which is possible if one doesn't already has $\xi_1^2 = 0$, $\xi_2^2 = 0$. Hence we are reduced to the case where $\xi_1^2 = 0$ and $\xi_2^2 = 0$.

(iv) The matrix $J$ has the form $\begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & E \end{pmatrix}$ with moreover $\xi_2^2 = -\xi_1^1$ and $\xi_4^4 = -\xi_3^3$. Consider $\Phi_\dagger = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ and ($\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$) $\Phi = \mathrm{diag}\left(\Phi_\dagger, \begin{pmatrix} H & -K \\ K & H \end{pmatrix}\right)$ where $U = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ , $V = \begin{pmatrix} \gamma & -\delta \\ \delta & \gamma \end{pmatrix}$ , and $H + iK = (\alpha - i\beta)(\gamma + i\delta)$. Then $\Phi \in \mathrm{Aut}(M5)$. Now

$$(\Phi_\dagger)^{-1} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \Phi_\dagger = \begin{pmatrix} U^{-1} & 0 \\ 0 & V^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} U^{-1}AU & 0 \\ 0 & V^{-1}D \end{pmatrix}.$$

One has

$$U^{-1}AU = \frac{1}{\alpha^2 + \beta^2} \begin{pmatrix} {\xi'}_1^1 & {\xi'}_2^1 \\ {\xi'}_1^2 & {\xi'}_2^2 \end{pmatrix},$$

with

$$\begin{aligned} {\xi'}_1^1 &= \alpha^2\xi_1^1 + \alpha\beta(\xi_2^1 + \xi_1^2) + \beta^2\xi_2^2 \\ {\xi'}_2^2 &= \alpha^2\xi_2^2 - \alpha\beta(\xi_2^1 + \xi_1^2) + \beta^2\xi_1^1. \end{aligned}$$

Since $\xi_1^1 = -\xi_2^2$, one has ${\xi'}_1^1 = -{\xi'}_2^2$. Then the discriminant of the equation ${\xi'}_2^2 = 0$ is the sum of two squares, hence there exist $\alpha, \beta \in \mathbb{R}$ with $\beta = 1$ such that ${\xi'}_1^1 = 0$ and ${\xi'}_2^2 = 0$. Then

$$U^{-1}AU = \frac{1}{\alpha^2+\beta^2}\begin{pmatrix} 0 & {\xi'}_2^1 \\ {\xi'}_1^2 & 0 \end{pmatrix}.$$

Similarly, there exist $\gamma, \delta \in \mathbb{R}$ with $\delta = 1$ such that

$$V^{-1}DV = \frac{1}{\gamma^2+\delta^2}\begin{pmatrix} 0 & {\xi'}_4^3 \\ {\xi'}_3^4 & 0 \end{pmatrix}.$$

Hence we are reduced to the case where $\xi_1^1 = \xi_3^3 = 0$. $\square$

### 13.4.3 Case 1.

In the present case 1, we suppose that $B$ and $C$ in (89) are not both 0. Then by equivalence (Lemma 2 (i)), we may suppose $B \neq 0$. Since $B \neq 0$, by equivalence (Lemma 2 (ii)), we may assume $\xi_3^2 = 1$, $\xi_4^2 = 0$. Again by equivalence (Lemma 2 (iii)), we may assume without altering $B$ that $\xi_1^2 = 0$, $\xi_2^2 = 0$. Then solving all equations, we finally get the matrix

$$J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6) = \begin{pmatrix} a & -\xi_5^6\xi_4^1 + \xi_5^6 - \xi_5^5\xi_3^1 & \xi_3^1 & \xi_4^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ b & c & d & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5^5 & -({\xi_5^5}^2+1)/\xi_5^6 \\ 0 & 0 & 0 & 0 & \xi_5^6 & -\xi_5^5 \end{pmatrix} \tag{90}$$

where :
$a = J_1^1 = (({\xi_5^5}^2+1)\xi_3^1 + (\xi_4^1-1)\xi_5^6\xi_5^5)/\xi_5^6$; $b = J_1^4 = -\frac{1+a^2}{\xi_4^1}$; $c = J_2^4 = ((\xi_5^6\xi_5^5(\xi_4^1-1)^2 + 2{\xi_5^5}^2\xi_3^1(\xi_4^1-1) + \xi_4^1\xi_3^1)\xi_5^6 + ({\xi_5^5}^2+1)\xi_5^5{\xi_3^1}^2)/(\xi_5^6\xi_4^1)$; $d = J_3^4 = ((\xi_5^6(\xi_4^1-1) + \xi_5^5\xi_3^1(2-\xi_4^1))\xi_5^6 - ({\xi_5^5}^2+1){\xi_3^1}^2)/(\xi_5^6\xi_4^1)$; and the parameters are subject to the condition

$$\xi_4^1\xi_5^6 \neq 0. \tag{91}$$

Note also the following formulae from $J^2 = -1$ :

$$d = \frac{1}{\xi_4^1}\left(\xi_5^6(\xi_4^1-1) + \xi_3^1(\xi_5^5 - a)\right) \quad , \quad c = ad + (1+a^2)\,\frac{\xi_3^1}{\xi_4^1}. \tag{92}$$

Commutation relations of $\mathfrak{m}$ : $[\tilde{x}_1, \tilde{x}_2] = a\tilde{x}_5 + d\tilde{x}_6$; $[\tilde{x}_1, \tilde{x}_3] = (b+1)\tilde{x}_5 - c\tilde{x}_6$; $[\tilde{x}_2, \tilde{x}_3] = \frac{1}{\xi_5^6\xi_4^1}\left((\xi_5^6(\xi_4^1-1) + \xi_5^5\xi_3^1)^2 + {\xi_3^1}^2\right)(\xi_5^5\tilde{x}_5 + \xi_5^6\tilde{x}_6)$; $[\tilde{x}_2, \tilde{x}_4] = (1-\xi_4^1)\tilde{x}_5 + \xi_3^1\tilde{x}_6$; $[\tilde{x}_3, \tilde{x}_4] = a\tilde{x}_5 + \left(\xi_5^6(\xi_4^1-1) + \xi_5^5\xi_3^1\right)\tilde{x}_6$.

$\mathfrak{m}$ is abelian if and only if

$$\xi_3^1 = 0 \quad , \quad \xi_4^1 = 1. \tag{93}$$

Suppose now that (93) holds, and denote $\alpha = \xi_5^5, \beta = \xi_5^6 \neq 0$, $J(\alpha, \beta) = J(0, 1, \xi_5^5, \xi_5^6)$. Then :

$$J(\alpha,\beta) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\frac{\alpha^2+1}{\beta} \\ 0 & 0 & 0 & 0 & \beta & -\alpha \end{pmatrix} \tag{94}$$

where $\beta \neq 0$. A direct study shows that any $J(\alpha, \beta)$ is equivalent to a unique $J(0, \gamma)$, $0 < \gamma \leqslant 1$, and any two such $J(0, \gamma)$, $0 < \gamma \leqslant 1$ are not equivalent unless $\gamma = \gamma'$.

### 13.4.4

Let $J_1 = J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ and $J_2 = J(\eta_3^1, \eta_4^1, \eta_5^5, \eta_5^6)$ as in (90). It is clear that $J_1 \cong J_2$ if and only if there exists a matrix $\Phi$ such that

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \end{pmatrix} \tag{95}$$

where $u = \pm 1$, $\Phi_\dagger, H, K$ are defined in Lemma 1, $H^2 + K^2 \neq 0$ and $J_2 = \Phi^{-1} J_1 \Phi$. This equation implies that

$$\begin{pmatrix} \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_5^6 & -\xi_5^5 \end{pmatrix} \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} = \begin{pmatrix} H & Ku \\ K & -Hu \end{pmatrix} \begin{pmatrix} \eta_5^5 & -\frac{{\eta_5^5}^2+1}{\eta_5^6} \\ \eta_5^6 & -\eta_5^5 \end{pmatrix}. \tag{96}$$

Equation (96) has non trivial solutions in $H, K$ if and only if the following condition holds :

$$\frac{{\eta_5^5}^2 + {\eta_5^6}^2 + 1}{\eta_5^6} = -u \frac{{\xi_5^5}^2 + {\xi_5^6}^2 + 1}{\xi_5^6}. \tag{97}$$

However, though that condition is also sufficient for the existence of $\Phi$ in (95) in the abelian case where $\xi_3^1 = \eta_3^1 = 0, \xi_4^1 = \eta_4^1 = 1$, it is no longer sufficient in the nonabelian case. For example, take $J(1,1,1,1)$ and $J(1,1,1,\eta_5^6)$, with $\eta_5^6 \neq 1$ : then (97) holds if only if $u = -1, \eta_5^6 = 2$ or $u = 1, \eta_5^6 = -2, -1$, and in neither case does equivalence occur; hence $J(1,1,1,1) \not\cong J(1,1,1,\eta_5^6)$ if $\eta_5^6 \neq 1$. However $J(1,1,0,1) \cong J(1,1,0,-1)$. One also has $J(\xi_3^1, 1, 0, 1)) \cong J(\eta_3^1, \eta_4^1, 0, 1) \Leftrightarrow \eta_3^1 = \pm\xi_3^1$ ; $\eta_4^1 = \xi_4^1$. In general, if $\eta_4^1 \neq \xi_4^1$, for $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ and $J(\eta_3^1, \eta_4^1, \eta_5^5, \eta_5^6)$ to be equivalent, it would be necessary that $P\xi_5^5 + Q = 0$ where $P, Q$ are certain huge polynomials in the other variables. We simply conjecture here that equivalence implies $\xi_4^1 = \eta_4^1$ and $\eta_3^1 = \pm\xi_3^1$, and leave open the equivalence problem in the nonabelian case.

### 13.4.5 Case 2.

We now suppose $B = C = 0$. Then necessarily $\xi_2^2 = -\xi_1^1, \xi_4^4 = -\xi_3^3$. By equivalence (Lemma 2 (iv)), we may suppose $\xi_1^1 = 0, \xi_3^3 = 0$.
**Case 2.1.** Suppose $\xi_5^5 = 0$. If $\xi_3^4\xi_1^2 \neq 1$ one gets then the matrix

$$J(\xi_1^2, \xi_3^4) = \mathrm{diag}\left( \begin{pmatrix} 0 & -1/\xi_1^2 \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\xi_3^4 \\ 1/\xi_3^4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & (\xi_3^4\xi_1^2 - 1)/(\xi_3^4 - \xi_1^2) \\ (-\xi_3^4 + \xi_1^2)/(\xi_3^4\xi_1^2 - 1) & 0 \end{pmatrix} \right) \tag{98}$$

with the conditions

$$\xi_1^2, \xi_3^4 \neq 0 \; ; \; \xi_3^4 \neq \xi_1^2, \frac{1}{\xi_1^2}. \tag{99}$$

If $\xi_3^4\xi_1^2 = 1$, we get

$$J = \mathrm{diag}\left( \begin{pmatrix} 0 & -1/\xi_1^2 \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\xi_1^2 \\ 1/\xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1/\xi_5^6 \\ \xi_5^6 & 0 \end{pmatrix} \right) \tag{100}$$

with the conditions

$$\xi_1^2 = \pm 1 \; ; \; \xi_5^6 \neq 0. \tag{101}$$

**Case 2.2.** Suppose $\xi_5^5 \neq 0$. Then we get:

$$J = \mathrm{diag}\left( \begin{pmatrix} 0 & -1/\xi_1^2 \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1/\xi_1^2 \\ \xi_1^2 & 0 \end{pmatrix}, \begin{pmatrix} \xi_5^5 & -({\xi_5^5}^2 + 1)/\xi_5^6 \\ \xi_5^6 & -\xi_5^5 \end{pmatrix} \right) \tag{102}$$

with the conditions

$$\xi_1^2 = \pm 1 \; ; \; \xi_5^5\xi_5^6 \neq 0. \tag{103}$$

### 13.4.6

Computing intertwining automorphisms, one can prove that for all $\xi_1^2, \xi_3^4, \eta_1^2, \eta_3^4 \in \mathbb{R}$ satisfying conditions (99), $J(\eta_1^2, \eta_3^4) \cong J(\xi_1^2, \xi_3^4)$ if and only if there exists $u = \pm 1$ such that one of the following is satisfied :

$$\left(\eta_1^2 = u\xi_1^2 \text{ or } \frac{u}{\xi_1^2}\right) \text{ and } \left(\eta_3^4 = u\xi_3^4 \text{ or } \frac{u}{\xi_3^4}\right)$$

or

$$\left(\eta_1^2 = u\xi_3^4 \text{ or } \frac{u}{\xi_3^4}\right) \text{ and } \left(\eta_3^4 = u\xi_1^2 \text{ or } \frac{u}{\xi_1^2}\right).$$

Conditions (99) are preserved by the transformations. For example, the canonical CS $J_0 = J(-1, 1)$ and its opposite $-J_0$ are equivalent. In fact, on has :

**Lemma 3.** *Let $\Omega$ denote the Aut$(M5)$-orbit of the canonical CS $J_0$. Then $\Omega$ is the 4-dimensional space comprised of the matrices*

$$J = \begin{pmatrix} 0 & -1/\xi_1^2 & 0 & 0 & 0 & 0 \\ \xi_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\xi_1^2 & 0 & 0 \\ 0 & 0 & -\xi_1^2 & 0 & 0 & 0 \\ \xi_2^6 & -\xi_1^6 & -\xi_4^6 & \xi_3^6 & 0 & 1/\xi_1^2 \\ \xi_1^6 & \xi_2^6 & \xi_3^6 & \xi_4^6 & -\xi_1^2 & 0 \end{pmatrix}$$

*where $\xi_1^2 = \pm 1, \xi_1^6, \xi_2^6, \xi_3^6, \xi_4^6 \in \mathbb{R}$.*

On the other hand, the $J$ in (100) appears simply as a limiting case when $\xi_5^5 \to 0$ of the structure $J(\xi_1^2, \xi_5^5, \xi_5^6)$ defined in (102) with $\xi_1^2 = \pm 1$ , $\xi_5^5 \xi_5^6 \neq 0$.

### 13.4.7

$M5$ is a complex algebra for the CS $J(\xi_1^2, \xi_3^4)$ in (98) if and only if $\xi_1^2 = -1, \xi_3^4 = 1$ or $\xi_1^2 = 1, \xi_3^4 = -1$, *i.e.* $J(\xi_1^2, \xi_3^4)$ is the canonical CS $J_0$ or its opposite respectively. Since $M5$ is not a complex algebra for the CS $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ in (90), the latter is not equivalent to $J_0$.

**Lemma 4.** *Suppose $J(\xi_1^2, \xi_3^4) \notin \{J_0, -J_0\}$. Then $J(\xi_1^2, \xi_3^4)$ is equivalent to some CS in case 1, i.e. there exist $\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6$ such that $J(\xi_1^2, \xi_3^4) \cong J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$.*

*Proof.* Take $\Phi_\dagger = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$ and

$$\Phi = \begin{pmatrix} \Phi_\dagger & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \quad \text{where} \quad S = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}). \tag{104}$$

Then $\Phi \in \text{Aut}(M5)$. Denote $J(\xi_1^2, \xi_3^4)_\dagger = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ where $A = \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} \\ \xi_1^2 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 & -\frac{1}{\xi_3^4} \\ \xi_3^4 & 0 \end{pmatrix}$. Then

$$(\Phi_\dagger)^{-1} J(\xi_1^2, \xi_3^4)_\dagger \, \Phi_\dagger = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AS - SD \\ 0 & D \end{pmatrix}.$$

Now

$$AS - SD = \begin{pmatrix} -\beta(\frac{1}{\xi_1^2} + \xi_3^4) & \alpha(\frac{1}{\xi_1^2} + \frac{1}{\xi_3^4}) \\ \alpha(\xi_1^2 + \xi_3^4) & \beta(\xi_1^2 + \frac{1}{\xi_3^4}) \end{pmatrix}.$$

The CS $\Phi^{-1} J(\xi_1^2, \xi_3^4) \, \Phi$ is of type 1 if $AS - SD \neq 0$. For this to hold, just choose $\alpha = 1, \beta = 0$ if $\xi_1^2 + \xi_3^4 \neq 0$ and $\alpha = 0, \beta = 1$ if $\xi_1^2 + \xi_3^4 = 0$, noting in this latter case that $\xi_1^2 + \frac{1}{\xi_3^4} = \frac{(\xi_1^2)^2 - 1}{\xi_1^2} \neq 0$ since $\xi_1^2 \neq \pm 1$ as $J(\xi_1^2, \xi_3^4) \notin \{J_0, -J_0\}$. $\square$

**Lemma 5.** *Let $J$ be either the CS defined in (102) with $\xi_1^2 = \pm 1$ , $\xi_5^5, \xi_5^6 \neq 0$ or the one in (100). Then $J \cong J(0, \beta)$ for some $\beta$ $(0 < \beta \leqslant 1)$, where $J(0, \beta)$ is defined in (94).*

*Proof.* In both cases, $\mathfrak{m}$ is abelian, and $J = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & E \end{pmatrix}$ where $A = \begin{pmatrix} 0 & -\frac{1}{\xi_1^2} \\ \xi_1^2 & 0 \end{pmatrix}$ , $E = \begin{pmatrix} \xi_5^5 & -\frac{{\xi_5^5}^2+1}{\xi_5^6} \\ \xi_5^6 & \xi_5^5 \end{pmatrix}$ with $\xi_5^6 \neq 0$, $\xi_1^2 = \pm 1$, $\xi_5^5 = 0$ in case (100) and $\xi_5^5 \neq 0$ in case (102). Take $\Phi, S$ as in (104). Denote $J_\dagger = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Then

$$(\Phi_\dagger)^{-1} J(\xi_1^2, \xi_3^4)_\dagger \, \Phi_\dagger = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AS - SA \\ 0 & A \end{pmatrix}.$$

Now $AS - SA = \begin{pmatrix} -2\beta\xi_1^2 & 2\alpha\xi_1^2 \\ 2\alpha\xi_1^2 & 2\beta\xi_1^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if we take $\alpha = \frac{\xi_1^2}{2}, \beta = 0$. Hence $\Phi^{-1} J \, \Phi$ is a CS of type 1, and we readily see that $J \cong J(0, 1, \xi_5^5, \xi_5^6)$, hence $J \cong J(0, \beta)$ for some $\beta$ , $0 < \beta \leqslant 1$. $\square$

To summarize, we have shown the following :

**Theorem 1.** *Any CS on the Lie algebra $M5$ is equivalent to either the canonical CS $J_0$ or some CS $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ defined in (90).*

### 13.4.8

$G_0$ is here the complex 3-dimensional Heisenberg group, considered as a real Lie group, *i.e.* the real Lie group comprised of the matrices

$$x = \begin{pmatrix} 1 & x^1 + iy^1 & x^3 + iy^3 \\ 0 & 1 & x^2 + iy^2 \\ 0 & 0 & 1 \end{pmatrix} \quad (x^k, y^k \in \mathbb{R} \quad \forall k = 1, 2, 3). \tag{105}$$

We here depart from the second kind coordinates to use the natural coordinates defined by (105). Then the matrix $x$ in (105) is $x = \exp\left(x^2 x_3 + y^2 x_4 + x^3 x_5 + y^3 x_6\right) \exp\left(x^1 x_1 - y^1 x_2\right)$, and

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x^1} \quad , \quad X_2 = -\frac{\partial}{\partial y^1} \\ X_3 &= \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + y^1 \frac{\partial}{\partial y^3} \quad , \quad X_4 = \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial y^3} \\ X_5 &= \frac{\partial}{\partial x^3} \quad , \quad X_6 = \frac{\partial}{\partial y^3}. \end{aligned}$$

### 13.4.9 Holomorphic functions for $J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$.

Let $G$ denote the group $G_0$ endowed with the left invariant structure of complex manifold defined by $J = J(\xi_3^1, \xi_4^1, \xi_5^5, \xi_5^6)$ in (90). One easily checks (with formulae (92)) that

$$\tilde{X}_6^- = -i \, \frac{1 - i\xi_5^5}{\xi_5^6} \, \tilde{X}_5^- \ ; \ \tilde{X}_1^- = -i \, \frac{1 + ia}{\xi_4^1} \, \tilde{X}_4^- \ ; \ i \, \tilde{X}_2^- - \tilde{X}_3^- = -i \left( id + \frac{(1 + ia)\xi_3^1}{\xi_4^1} \right) \tilde{X}_4^- .$$

Hence $H_{\mathbb{C}}(G) = \{ f \in C^\infty(G_0) \ ; \ \tilde{X}_j^- \, f = 0 \ \forall j = 1, 3, 5 \}$. Consider first the equation

$$\tilde{X}_5^- \, f = 0. \tag{106}$$

One has

$$\tilde{X}_5^- = X_5 + i(\xi_5^5 X_5 + \xi_5^6 X_6) = \frac{\partial}{\partial x^3} + i \left( \xi_5^5 \, \frac{\partial}{\partial x^3} + \xi_5^6 \, \frac{\partial}{\partial y^3} \right) = \frac{\partial}{\partial u^3} + i \, \frac{\partial}{\partial v^3} = 2 \, \frac{\partial}{\partial \overline{w^3}} \tag{107}$$

where $u^3 = x^3 - \frac{\xi_5^5}{\xi_5^6} \, y^3$ ; $v^3 = \frac{y^3}{\xi_5^6}$ ; $w^3 = u^3 + iv^3$. Equation (106) simply means that $f$ is holomorphic with respect to $w^3$. Consider now the equation

$$\tilde{X}_1^- \, f = 0. \tag{108}$$

One has

$$\tilde{X}_1^- = X_1 + i(aX_1 + bX_4) = \frac{\partial}{\partial x^1} + i\left(a\,\frac{\partial}{\partial x^1} + b\,\frac{\partial}{\partial y^2}\right) + ib\left(-y^1\,\frac{\partial}{\partial x^3} + x^1\,\frac{\partial}{\partial y^3}\right).$$

We suppose that $f$ satisfies equation (106), *i.e.* $f$ is holomorphic with respect to $w^3$. Hence

$$\frac{\partial f}{\partial x^3} = \frac{\partial f}{\partial u^3} = \frac{\partial f}{\partial w^3}$$

$$\frac{\partial f}{\partial y^3} = \frac{1}{\xi_5^6}\frac{\partial f}{\partial v^3} - \frac{\xi_5^5}{\xi_5^6}\frac{\partial f}{\partial u^3} = \frac{1}{\xi_5^6}(i - \xi_5^5)\frac{\partial f}{\partial w^3}.$$

Then equation (108) reads

$$\frac{\partial f}{\partial x^1} + i\left(a\,\frac{\partial f}{\partial x^1} + b\,\frac{\partial f}{\partial y^2}\right) - b\left(iy^1 + \frac{x^1}{\xi_5^6}(1 + i\xi_5^5)\right)\frac{\partial f}{\partial w^3} = 0$$

that is

$$2\,\frac{\partial f}{\partial \overline{w^1}} - \frac{b}{2}\,\left(2iy^1 + \frac{1 + i\xi_5^5}{\xi_5^6}((1 - ia)w^1 + (1 + ia)\overline{w^1})\right)\frac{\partial f}{\partial w^3} = 0 \tag{109}$$

where $u^1 = x^1 - \frac{a}{b}\,y^2$ ; $v^1 = \frac{y^2}{b}$ ; $w^1 = u^1 + iv^1$. Finally, we turn to the last equation

$$\tilde{X}_3^-\,f = 0. \tag{110}$$

One has

$$\begin{aligned}\tilde{X}_3^- &= i\xi_3^1 X_1 + iX_2 + X_3 + idX_4\\ &= i\xi_3^1\frac{\partial}{\partial x^1} - i\,\frac{\partial}{\partial y^1} + \left(\frac{\partial}{\partial x^2} + x^1\,\frac{\partial}{\partial x^3} + y^1\,\frac{\partial}{\partial y^3}\right) + id\,\left(\frac{\partial}{\partial y^2} - y^1\,\frac{\partial}{\partial x^3} + x^1\,\frac{\partial}{\partial y^3}\right).\end{aligned}$$

Since we suppose $f$ holomorphic with respect to $w^3$, equation (110) then reads

$$\frac{\partial f}{\partial x^2} - i\,\frac{\partial f}{\partial y^1} + i\,\xi_3^1\frac{\partial f}{\partial x^1} + id\,\frac{\partial f}{\partial y^2} + \left(x^1 - idy^1 + (y^1 + idx^1)\,\frac{i - \xi_5^5}{\xi_5^6}\right)\frac{\partial f}{\partial w^3} = 0$$

that is

$$\begin{aligned}2\,\frac{\partial f}{\partial \overline{w^2}} + \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\frac{\partial f}{\partial w^1} + \left(i\left(\xi_3^1 - \frac{ad}{b}\right) + \frac{d}{b}\right)\frac{\partial f}{\partial \overline{w^1}}&\\ + \frac{1}{2}\left[\left((1 - ia)w^1 + (1 + ia)\overline{w^1}\right)\left(1 - \frac{d(1 + i\xi_5^5)}{\xi_5^6}\right)\right.&\\ \left.-(w^2 - \overline{w^2})\left(-d + \frac{1 + i\xi_5^5}{\xi_5^6}\right)\right]\frac{\partial f}{\partial w^3} = 0 \quad (111)&\end{aligned}$$

where $w^2 = x^2 - iy^1$. Now equation (109) reads

$$2\,\frac{\partial f}{\partial \overline{w^1}} - \frac{b}{2}\,\left(-w^2 + \overline{w^2} + \frac{1 + i\xi_5^5}{\xi_5^6}((1 - ia)w^1 + (1 + ia)\overline{w^1})\right)\frac{\partial f}{\partial w^3} = 0. \tag{112}$$

From equations (107), (112), (111), one readily sees that the functions $\varphi^1$ and $\varphi^2$ defined by

$$\varphi^1 = 2w^1 - \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\overline{w^2} \tag{113}$$

$$\varphi^2 = w^2 \tag{114}$$

are holomorphic on $G$. We look for a holomorphic function which depends on $w^3$. For any $C^\infty$-function $\psi(w^1, w^2, \overline{w^2})$, *i.e.* $\psi$ doesn't depend on $w^3, \overline{w^3}, \overline{w^1}$, the following function $f_1$ is a solution of equations (106) and (112) :

$$f_1 = w^3 + \frac{b}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right] + \psi(w^1, w^2, \overline{w^2}). \quad (115)$$

We want to choose $\psi$ such that $f_1$ is a solution of (111) as well. First, we have :

$$\begin{aligned}
\frac{\partial f_1}{\partial w^1} &= \frac{b}{4\xi_5^6}(1-ia)(1+i\xi_5^5)\,\overline{w^1} + \frac{\partial \psi}{\partial w^1}\\
\frac{\partial f_1}{\partial \overline{w^1}} &= \frac{b}{4}\left(-w^2 + \overline{w^2} + \frac{1+i\xi_5^5}{\xi_5^6}((1-ia)w^1 + (1+ia)\overline{w^1})\right)\\
\frac{\partial f_1}{\partial \overline{w^2}} &= \frac{b}{4}\,\overline{w^1} + \frac{\partial \psi}{\partial \overline{w^2}}\\
\frac{\partial f_1}{\partial w^3} &= 1\,.
\end{aligned}$$

Introducing these values in (111) we find that $f_1$ is a solution to (111) if and only if

$$N\,\overline{w^1} - M(w^2 - \overline{w^2}) + \Lambda w^1 + \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\frac{\partial \psi}{\partial w^1} + 2\,\frac{\partial \psi}{\partial \overline{w^2}} = 0$$

where

$$\begin{aligned}
N &= \frac{1}{\xi_5^6}\left(ib\xi_3^1 - b\xi_3^1\xi_5^5 + b\xi_5^6 + (1+ia)\xi_5^6 - d(1+ia)(1+i\xi_5^5)\right)\\
M &= \frac{1}{4}(ib\xi_3^1 + (1-ia)d) + \frac{1}{2}\left(\frac{1}{\xi_5^6} - d + i\frac{\xi_5^5}{\xi_5^6}\right)\\
\Lambda &= \frac{(1+i\xi_5^5)(1-ia)}{4\xi_5^6}\left(ib\xi_3^1 + (1-ia)d\right) + \frac{1-ia}{2}\left(1 - \frac{d(1+i\xi_5^5)}{\xi_5^6}\right).
\end{aligned}$$

A computation shows that $N$ is actually equal to 0. Hence $f_1$ is a solution to (111) if and only if

$$-M(w^2 - \overline{w^2}) + \Lambda w^1 + \left(i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}\right)\frac{\partial \psi}{\partial w^1} + 2\,\frac{\partial \psi}{\partial \overline{w^2}} = 0. \quad (116)$$

Note that

$$i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b} = 0 \Leftrightarrow \xi_3^1 = d = 0 \Leftrightarrow \xi_3^1 = 0, \xi_4^1 = 1.$$

Hence, in the nonabelian case where one doesn't have simulteanously $\xi_3^1 = 0, \xi_4^1 = 1$, for (116) to hold, it is sufficient to have

$$\begin{aligned}
\frac{\partial \psi}{\partial w^1} &= -\frac{\Lambda}{i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}}\,w^1\\
\frac{\partial \psi}{\partial \overline{w^2}} &= \frac{M}{2}(w^2 - \overline{w^2})
\end{aligned}$$

which gives a solution

$$\psi = -\frac{\Lambda}{i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}}\,\frac{(w^1)^2}{2} + \frac{M}{2}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right).$$

In the abelian case, one can take

$$\psi = -\frac{\Lambda}{2}\,w^1\overline{w^2} + \frac{M}{2}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right).$$

We finally get the holomorphic function $f_1$ : in the nonabelian case,

$$f_1 = w^3 + \frac{b}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right] \\ - \frac{\Lambda}{i\left(\xi_3^1 - \frac{ad}{b}\right) - \frac{d}{b}}\,\frac{(w^1)^2}{2} + \frac{M}{2}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right). \quad (117)$$

In the abelian case,

$$f_1 = w^3 + \frac{b}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1+i\xi_5^5}{\xi_5^6}\left((1-ia)w^1\overline{w^1} + (1+ia)\frac{(\overline{w^1})^2}{2}\right)\right] \\ - \frac{\Lambda}{2}\,w^1\overline{w^2} + \frac{M}{2}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right). \quad (118)$$

Note that in the abelian case, one can take $\xi_5^5 = 0, \xi_5^6 = \beta$ , $0 < \beta \leqslant 1$, and then $a = c = d = 0$, $b = -1, M = \frac{1}{2\beta}, \Lambda = \frac{1}{2}$, hence

$$f_1 = w^3 - \frac{1}{4}\left[-w^2\,\overline{w^1} + \overline{w^2}\,\overline{w^1} + \frac{1}{\beta}\left(w^1\overline{w^1} + \frac{(\overline{w^1})^2}{2}\right)\right] \\ - \frac{1}{4}w^1\overline{w^2} + \frac{1}{4\beta}\left(w^2\overline{w^2} - \frac{(\overline{w^2})^2}{2}\right). \quad (119)$$

In both abelian and nonabelian cases, let $F : G \to \mathbb{C}^3$ defined by $F = (\varphi^1, \varphi^2, \varphi^3)$ where $\varphi^1, \varphi^2$ are defined in (113), (114) and $\varphi^3 = f_1$. $F$ is a global chart on $G$. We determine now how the multiplication of $G$ looks like in the chart $F$. Recall first the formulae :

$$w^1 = \left(x^1 - \frac{a}{b}y^2\right) + i\frac{y^2}{b} \quad , \quad w^2 = x^2 - iy^1 \quad , \quad w^3 = \left(x^3 - \frac{\xi_5^5}{\xi_5^6}\right) + i\,\frac{y^3}{\xi_5^6}. \quad (120)$$

Let $a, x \in G$ with respective canonical coordinates $(x^1, y^1, x^2, y^2, x^3, y^3), (\alpha^1, \beta^1, \alpha^2, \beta^2, \alpha^3, \beta^3)$ as in (105). With obvious notations, by matrix multiplication and (120) one gets

$$w^1_{\alpha x} = w^1_\alpha + w^1_x \quad , \quad w^2_{\alpha x} = w^2_\alpha + w^2_x \quad , \quad w^3_{\alpha x} = w^3_\alpha + w^3_x + \chi(\alpha, x)$$

where $\chi(\alpha, x) = \alpha^1 x^2 - \beta^1 y^2 + \frac{i-\xi_5^5}{\xi_5^6}\left(\alpha^1 y^2 + \beta^1 x^2\right)$. Then from (113), (114) :

$$\varphi^1_{\alpha x} = \varphi^1_\alpha + \varphi^1_x \quad , \quad \varphi^2_{\alpha x} = \varphi^2_\alpha + \varphi^2_x.$$

To get $\varphi^3_{\alpha x}$, we just make the substitutions $w^1 \to w^1_\alpha + w^1_x$, $w^2 \to w^2_\alpha + w^2_x$, $w^3 \to w^3_\alpha + w^3_x + \chi(\alpha, x)$ in (117) (we consider here the nonabelian case). Now, let $\Delta = \varphi^3_{\alpha x} - \varphi^3_\alpha - \varphi^3_x$. Computations give:

$$\Delta = \frac{1}{8\xi_5^6}\left(C_1\varphi^1_x + C_2\xi_5^6\varphi^2_x\right)$$

where

$$C_1 = b(1+i\xi_5^5)\overline{\varphi^1_\alpha} + (b-1+ia)\xi_5^6\overline{\varphi^2_\alpha} + \frac{b(a+i)\xi_5^6}{b\xi_3^1 - d(a-i)}\varphi^1_\alpha + \left((\xi_5^5 - i)(b\xi_3^1 - da - id) - b\xi_5^6\right)\varphi^2_\alpha \quad ,$$

$$C_2 = (1-b+ia)\overline{\varphi^1_\alpha} \\ + \left(\frac{4(1+i\xi_5^5)}{\xi_5^6} + (a+i(b+1))\xi_3^1 - \frac{d}{b}(1+a^2) - d(1+ia)\right)\overline{\varphi^2_\alpha} \\ + (1-ai)\varphi^1_\alpha \\ + \left(-\frac{2(1+i\xi_5^5)}{\xi_5^6} + (a+i(b-1))\xi_3^1 - \frac{d}{b}(1+a^2) + d(1-ia)\right)\varphi^2_\alpha.$$

For example, in the case of $J(1,1,1,1)$,

$$\Delta = \left( -\frac{3-i}{4}\,\overline{\varphi^2_\alpha} - \frac{5+5i}{8}\,\overline{\varphi^1_\alpha} + \frac{3+4i}{8}\,\varphi^2_\alpha + \frac{7+i}{16}\,\varphi^1_\alpha \right) \varphi^1_x + \left( \frac{3+i}{4}\,\overline{\varphi^2_\alpha} + \frac{3+i}{4}\,\overline{\varphi^1_\alpha} - \frac{1+3i}{4}\,\varphi^2_\alpha + \frac{1-2i}{8}\,\varphi^1_\alpha \right) \varphi^2_x \,.$$

Finally, in the abelian case $J(0,1,0,\beta)$, one has to use (119) and one gets :

$$\Delta = -\left( \frac{1}{4}\,\overline{\varphi^2_\alpha} + \frac{1}{8\beta}\,\overline{\varphi^1_\alpha} - \frac{1}{8}\,\varphi^2_\alpha + \frac{1}{16\beta}\,\varphi^1_\alpha \right) \varphi^1_x + \left( \frac{1}{2\beta}\,\overline{\varphi^2_\alpha} + \frac{1}{4}\,\overline{\varphi^1_\alpha} - \frac{1}{4\beta}\,\varphi^2_\alpha + \frac{1}{8}\,\varphi^1_\alpha \right) \varphi^2_x \,.$$

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